

Theory of adiabatic quantum control in the presence of cavity-photon shot noise

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March 2014

A thesis submitted to McGill University in partial fulfillment of
the requirements of the degree of M.Sc.

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À mes parents, à Marie-Michelle et à mon oncle Claude.

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Abstract

Many areas of physics rely upon adiabatic state transfer protocols, allowing a quantum state to be moved between different physical systems for storage and retrieval or state manipulation. However, these state-transfer protocols suffer from dephasing and dissipation. In this thesis we go beyond the standard open-systems treatment of quantum dissipation allowing us to consider non-Markovian environments. We use adiabatic perturbation theory in order to give analytic descriptions for various quantum state-transfer protocols. The leading-order corrections will give rise to additional terms adding to the geometric phase preventing us from achieving a perfect fidelity. We obtain analytical descriptions for the effects of the geometric phase in non-Markovian regimes. The Markovian regime is usually treated by solving a standard Bloch-Redfield master equation, while in the non-Markovian regime, we perform a secular approximation allowing us to obtain a solution to the density matrix without solving master equations. This solution contains all the relevant phase information for our state-transfer protocol. After developing the general theoretical tools, we apply our methods to adiabatic state transfer between a four-level atom in a driven cavity. We explicitly consider dephasing effects due to unavoidable photon shot noise and give a protocol for performing a phase gate. These results will be useful to ongoing experiments in circuit quantum electrodynamics (QED) systems.

Résumé

Plusieurs domaines de la physique se font sur des protocoles adiabatiques de transfert d'état, permettant un état quantique de se déplacer entre différents systèmes physiques afin de les emmagasiner, les récupérer ou de les manipuler. Par contre, ces protocoles de transferts d'état souffrent du phénomène de déphasage et de dissipation. Dans ce mémoire, nous allons au-delà des traitements standards de systèmes ouverts décrivant la dissipation quantique nous permettant de considérer des environnements non-Markoviens. Nous utilisons la théorie de perturbation adiabatique afin de donner une description analytique à plusieurs protocoles de transfert quantique. Les corrections d'ordres principaux donnent lieu à une phase géométrique de système ouvert, ce qui nous empêche d'atteindre une fidélité parfaite. Nous obtenons une description analytique pour les effets de phase géométrique dans des régimes non-Markovien. Les régimes Markovien sont habituellement traités en trouvant une solution à une équation de maîtresse Bloch-Redfield, tandis que pour les régimes non-Markovien, nous allons performer une approximation séculaire nous permettant d'obtenir une solution exacte de la matrice de densité qui contient toute l'information de phase pertinente pour notre protocole de transfert d'état. Après avoir développé les outils théoriques généraux, nous allons appliquer nos méthodes à un transfert d'état adiabatique entre les états d'un atome à quatre niveaux dans une cavité entraînée. Nous considérons explicitement les effets de déphasage dus au bruit photonique inévitable. Ces résultats seront utiles pour des expériences en cours dans des systèmes de circuits QED.

Acknowledgments

I would like to sincerely thank both of my supervisors, Aash Clerk and Bill Coish for their continued support and guidance throughout my masters degree. Your never ending patience and very generous office hours have proven to be extremely valuable in making progress during my masters degree. I am also very grateful to have had the chance to work with two extremely talented and visionary physicists. This has without a doubt been the most inspiring part of my masters degree. Both of you have been very helpful in making me become a more independent researcher which are skills that I will carry with me for the rest of my life.

I would like to give many thanks to Tami Pereg-Barnea for sharing much of her research ideas and opening doors into a new world of physics that I would have never thought I would be interested in.

I would like to thank Benjamin D’Anjou, Félix Beaudoin, Marc-Antoine Lemonde, Saeed Khan, Judy Wang, Abhishek Kumar and Pawel Mazurek for your guidance and many scientific discussions that have not only been essential but very enlightening throughout my masters degree. I would also like to thank Aaron Farrell, Pericles Philippopoulos and Ben Levitan for your many collaborations and scientific ideas.

I would like to thank Wei Chen, Stefano Chesi, Anja Metelmann and Nicolas Didier for their very sound advice and help in guiding me throughout my masters degree.

I owe a very large thank you to Marius Oltean. Your constant presence and support has been extremely valuable. Your help in teaching me LaTeX and numeric’s has been essential on many occasions and I could not have done it without you. I would also like to thank you for our many deep scientific dicussions about the very foundation of physics and especially quantum mechanics. There are very few people who worry about these types of questions

but they are of uttermost importance. I am very grateful for having had a friend like you and it has certainly opened my eyes to some of the most fascinating features that physics has to offer.

J'aimerais remercier mon oncle Claude pour son hébergement, son support constant et ses conseils extrêmement pratique. Merci aussi pour m'avoir guidé à travers la ville de Montréal et pour les délicieux repas.

Finally, I owe my infinite gratitude to my parents and sister for their continued support throughout my time at McGill. Thank you for taking the time to put up with me during hard times and for your continued moral support. Also, thank you for always looking out for me and making sure sure that I always focus on the most important things.

CHAPTER 1

Introduction

Quantum state transfer arises when one would like to transfer a quantum state from one system to another with the highest possible fidelity. There are many state-transfer protocols that are of practical significance. For example, if one is interested in building a secure quantum system for private communication or for long-distance quantum communication, it is necessary to transfer information between quantum-memory atoms and photons [1, 2, 3, 4, 5, 6]. In other situations, it is crucial to be able to preserve a quantum state for as long as possible in order to trade states back and fourth between an ensemble of atoms and photons [7, 8, 9, 10, 11, 12, 13]. Quantum state transfer can also lead to striking results. For instance, when applied to quantum-logic clocks, they become so precise that they are able to keep time to within 1 second every 3.7 billion years [14]. In reality, when one performs a state-transfer protocol, the quantum systems under consideration are never completely isolated from an exterior environment. Consequently, regardless of which state-transfer protocol is being employed, one will always be faced with the challenge of fighting decoherence.

During the early days of quantum mechanics, the adiabatic theorem was a perturbative tool that was developed to deal with slowly-varying time-dependent Hamiltonians [15, 16]. The Hamiltonian $H(t)$ must vary slowly compared to internal time scales of the system quantified by a dimensionless parameter A . For example, the parameter A for a spin-half coupled to a time-dependent magnetic field would be Bt_p where t_p is the total time it takes the magnetic field to complete a closed loop trajectory in 3-D space [29]. To leading order in adiabatic perturbation theory, the instantaneous eigenstates of the Hamiltonian acquire a geometric phase (along with its usual dynamical phase). The term “geometric phase”

comes from the fact that this phase depends on the path traversed by the system in Hilbert space but not on the time to traverse the path [17]. Recently, it has been shown that in the framework of quantum information and quantum computation, the geometric phase is robust in the presence of some sources of decoherence [18, 19, 20, 21]. Thus, it may be useful to take advantage of this robustness to perform a phase gate in the presence of a dissipative quantum system. On the other hand, corrections to the geometric phase in the presence of a dissipative environment will be the leading source preventing one from doing a state-transfer protocol with perfect fidelity. Given these considerations, it is of paramount importance to study geometric phases for open quantum systems. Many authors [22, 23, 24, 25, 26, 27, 28, 29] have considered geometric phases for systems coupled to classical or quantum environments. The authors of [29] consider a spin-1/2 system which is both subject to a slowly-varying magnetic field and weakly coupled to a dissipative environment (which the authors choose to be quantum). The authors obtained a modification to the closed-system geometric phase due to the coupling to the environment and found that this phase is also geometrical. However, the authors' results are limited by the fact that they consider only weak dissipation and an environment giving rise to Markovian evolution. In this thesis, we develop a general theory that gives rise to non-Markovian evolution by performing a secular approximation following the criteria of [30]. As was shown in [30], performing a secular approximation on time-dependent systems can often be problematic and must be done with care. Consequently, we will go into some detail in order to ensure that the secular approximation is done correctly. Given that our main motivation is to perform a phase gate under a quantum state-transfer protocol while minimizing dephasing effects, we will develop our theory under the framework of a Stimulated Raman Adiabatic Passage (STIRAP) applied to a tripod system (Fig. 2.2.1). A STIRAP system is an adiabatic process that is efficient at transferring population in a three-level system [31, 32, 33, 34, 35, 36, 37, 38]. When applied to a tripod system, one now deals with four levels (instead of three) allowing the possibility to transfer population amongst the two ground state levels (the levels spacing $|g_1\rangle$ and $|g_2\rangle$ in Fig. 2.2.1). Our task will be twofold. First we will perform a state-transfer protocol in the presence of a dissipative environment. Secondly,

we will perform a phase-gate state-transfer protocol by transferring quantum information between the four-level system the presence of a dissipative environment. The fourth level (the state $|0\rangle$ in Fig. 2.2.1) in our tripod system will act as a spectator state meaning that no driving terms will couple it to an excited state (unlike the two other levels of the tripod system which are coupled to an excited state via time-dependent classical fields). In this case, the spectator state will not acquire a geometric phase. However, the other instantaneous eigenstates of our system will acquire an environment-induced (as well as a closed-system) geometric phase. It will thus be crucial to understand how non-Markovian environments modify the closed-system geometric phase.

In chapter 2 we give an overview of adiabatic perturbation theory and of the dynamics of the geometric phase. We will then apply adiabatic perturbation theory to describe the closed-system four-level adiabatic state-transfer protocol described above. We will also show how all the phase information is encoded in the off-diagonal components of the density matrix and how our state-transfer protocol can be described in the language of density matrices. We will conclude chapter 2 by giving an overview of cavity-photon shot noise. In chapter 3, we develop a theory allowing us to calculate dephasing effects in our state-transfer protocol for the case where the system is coupled to a quantum dissipative bath. We will do this by performing a secular approximation in a superadiabatic basis allowing us to obtain the relevant off-diagonal component of the density matrix. We will conclude chapter 3 by applying our theory to a system of independent bosonic modes coupled to our four-level system. In chapter 4, we will apply our methods to adiabatic state transfer of a four-level system in a driven cavity. We will explicitly consider dephasing effects due to unavoidable photon shot noise. Once the dephasing effects have been accounted for, we will show how it is possible to perform a phase gate by choosing specific functional forms for the phase difference between the two laser fields driving the cavity. These results will be useful to ongoing experiments in circuit QED systems.

CHAPTER 2

Adiabatic evolution and noise

2.1. Adiabatic approximation and the Berry's phase

When dealing with physical systems where the Hamiltonian takes on an explicit time dependence, it is often very difficult to solve the Schroedinger equation exactly. One must often approach the problem using perturbative techniques to have some hope of getting analytical results. The adiabatic approximation is very useful for cases where the Hamiltonian changes slowly in time. However, at this stage, one must correctly define what we mean by changing “slowly in time”. To address this issue on analytical grounds, we can start by looking at the energy eigenvalues of the Hamiltonian. For a fixed value of time, it is always possible to obtain the energy spectrum of the Hamiltonian which depends on a set of parameters. Consequently, when we say that the Hamiltonian changes slowly in time, we mean that the set of parameters change on a time scale T that is much larger than $2\pi/E_{ab}$ where E_{ab} are the energy differences between two levels. To describe this mechanism mathematically, we follow closely the treatment given in [39]. The first step is to find the instantaneous eigenstates of the Hamiltonian. Since the eigenstates are defined for each moment in time, we can represent them as

$$\hat{H}(t) |\phi_n(t)\rangle = E(t) |\phi_n(t)\rangle. \quad (2.1.1)$$

Here, $|\phi_n(t)\rangle$ correspond to the instantaneous eigenstates of the Hamiltonian and $E(t)$ are the corresponding instantaneous eigenenergies. We can write the usual Schroedinger equation as (setting $\hbar = 1$)

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle. \quad (2.1.2)$$

The trick is to expand the wave function satisfying the Schroedinger equation as

$$|\psi(t)\rangle = \sum_n c_n(t) e^{i\theta_n(t)} |\phi_n(t)\rangle, \quad (2.1.3)$$

where we define

$$\theta_n(t) \equiv - \int_0^t E_n(t') dt'. \quad (2.1.4)$$

Note that in (2.1.3) the coefficients $c_n(t)$ are taken to be real. The phase $\theta_n(t)$ is referred to as the dynamical phase. Next we substitute the expansion (2.1.3) into (2.1.2) and use the property (2.1.1). After a little bit of algebra and using the orthonormality property of the instantaneous eigenstates, we find the following differential equation for the expansion coefficients

$$\dot{c}_m(t) = - \sum_n c_n(t) e^{i[\theta_n(t) - \theta_m(t)]} \langle \phi_m(t) | [\partial_t | \phi_n(t) \rangle]. \quad (2.1.5)$$

Notice the inner product $\langle \phi_m(t) | [\partial_t | \phi_n(t) \rangle]$ appearing in equation (2.1.5). Since the Hamiltonian is time dependent, this quantity will not vanish in general. Thus, we must find a way to calculate it. This can be done by going back to the eigenvalue equation (2.1.1) and taking a time derivative on both sides of it. If we restrict ourselves to the case where $m \neq n$, we find that

$$\langle \phi_m(t) | \dot{H}(t) | \phi_n(t) \rangle = [E_n(t) - E_m(t)] \langle \phi_m(t) | [\partial_t | \phi_n(t) \rangle]. \quad (2.1.6)$$

We can replace (2.1.6) into (2.1.5) for the term where $m \neq n$ and this term can thus be written as

$$\dot{c}_m(t) = -c_m(t) \langle \phi_m(t) | [\partial_t | \phi_m(t) \rangle] - \sum_{n \neq m} c_n(t) e^{i[\theta_n(t) - \theta_m(t)]} \frac{\langle \phi_m(t) | \dot{H}(t) | \phi_n(t) \rangle}{E_n - E_m}. \quad (2.1.7)$$

So far everything is exact. Due to the second term in (2.1.7), as the system evolves in time, the states $|\phi_n(t)\rangle$ (with $n \neq m$) will mix with $|\phi_m(t)\rangle$. The adiabatic approximation consists of neglecting the second term in (2.1.7). In other words, we must require that

$$\left| \frac{\langle \phi_m(t) | \dot{H}(t) | \phi_n(t) \rangle}{E_n - E_m} \right| \ll |\langle \phi_m(t) | [\partial_t | \phi_m(t) \rangle]|. \quad (2.1.8)$$

If we identify the term on the left of (2.1.8) as $1/\tau$, where τ describes the characteristic time scale for changes in the Hamiltonian and the term on the right as E_m (natural frequency of the state-phase factor), then we require that $\tau \gg 1/E_m$ [39]. In other words, the characteristic time scale for changes in the Hamiltonian must be much larger than the inverse natural frequency of the state-phase factor. However, we must be careful in using (2.1.8) to establish the validity of the adiabatic approximation. To see this, we consider the following argument. Consider an equation of motion of the form

$$\dot{c}_m(t) = \delta c_m^0(t) + \delta c_m^\nu(t), \quad (2.1.9)$$

with the condition

$$|\delta c_m^0(t)| \gg |\delta c_m^\nu(t)|, \quad (2.1.10)$$

(here the coefficients $\delta c_m^0(t)$ and $\delta c_m^\nu(t)$ are time-dependent functions analogous to the first and second term on the right hand side of (2.1.7)). We have to be very careful when we say that we can neglect the second term under the condition (2.1.10). If we integrate (2.1.9) we get

$$c_m(t) = c_m(0) + \int_0^t \delta c_m^0(t') dt' + \int_0^t \delta c_m^\nu(t') dt'. \quad (2.1.11)$$

After a time scale \tilde{t} the second term in (2.1.11) could be of order one and thus could no longer be neglected. The important information to gather out of this argument is that we can always use (2.1.8) as a criterion for the adiabatic approximation as long as we are within a time scale which guarantees that the term

$$\int_0^t \sum_n c_n(t') e^{i[\theta_n(t') - \theta_m(t')]} \frac{\langle \phi_m(t') | \dot{\hat{H}}(t') | \phi_n(t') \rangle}{[E_n(t') - E_m(t')]} dt' \lesssim 1, \quad (2.1.12)$$

is below of order one. Above this time scale we can no longer use adiabatic perturbation theory with confidence. Assuming we are within a time scale such that (2.1.8) is valid, (2.1.7) will simplify to

$$c_n(t) \cong e^{i\gamma_n(t)} c_n(0), \quad (2.1.13)$$

where

$$\gamma_n(t) \equiv i \int_0^t \langle \phi_n(t') | [\partial_t | \phi_n(t') \rangle] dt'. \quad (2.1.14)$$

The function $\gamma_n(t)$ is called Berry's phase and it satisfies some nice properties. As a first note, $\gamma_n(t)$ is real which can be seen from the fact that differentiating the inner product of $|\phi_n(t)\rangle$ with itself gives zero so that

$$\langle \phi_n(t) | [\partial_t | \phi_n(t) \rangle] = -(\langle \phi_n(t') | [\partial_t | \phi_n(t') \rangle])^*. \quad (2.1.15)$$

The integrand of (2.1.14) is then purely imaginary. Coming back to the state (2.1.3) and using (2.1.13), we find that

$$|\psi_n(t)\rangle = e^{i\gamma_n(t) + i\theta_n(t)} |\phi_n(t)\rangle. \quad (2.1.16)$$

We can thus conclude that in the adiabatic approximation, the general solution to the Schroedinger equation is given by a linear combination of instantaneous eigenstates modulated by a dynamical plus geometric phase. As will be seen below, Berry's phase (also known as the geometric phase) plays a crucial role for systems that are cyclic in time. Note that the derivation of the geometric phase has been for closed quantum systems. In chapter 3 we will consider quantum systems interacting with an environment. The interaction with the environment will induce corrections to the geometric phase and part of the goal of chapter 3 will be to develop methods that will allow us to calculate and interpret these corrections.

2.2. Quantum state transfer protocol

In this section we will give a quantitative description of the physical system describing how information will be transferred between two qubit states. Throughout this thesis we will often perform unitary transformations on the Hamiltonian allowing us to go into a rotating frame. For instance, this will be essential when writing our Hamiltonian in a superadiabatic basis allowing us to perform a secular approximation [30]. To describe the transformation of

the Hamiltonian when going into a rotating frame, we start with the Schroedinger equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle. \quad (2.2.1)$$

Next we apply a unitary transformation to the Schroedinger picture eigenstates

$$|\tilde{\psi}(t)\rangle = \hat{U}(t) |\psi(t)\rangle. \quad (2.2.2)$$

At this stage we would like to write down a Schroedinger equation for the transformed eigenstates $|\tilde{\psi}(t)\rangle$. We can use (2.2.1) by writing $|\psi(t)\rangle = \hat{U}(t)^{-1} |\tilde{\psi}(t)\rangle$ (note also the time dependence in the unitary operator $\hat{U}(t)$). We then have

$$i \frac{\partial}{\partial t} \left(\hat{U}(t)^{-1} |\tilde{\psi}(t)\rangle \right) = \hat{H}(t) \left(\hat{U}(t)^{-1} |\tilde{\psi}(t)\rangle \right). \quad (2.2.3)$$

Using the fact that $\hat{U}(t)$ is time dependent (2.2.3) becomes

$$\begin{aligned} \left(\frac{\partial \hat{U}(t)^{-1}}{\partial t} |\tilde{\psi}(t)\rangle + \hat{U}(t)^{-1} \frac{\partial}{\partial t} |\tilde{\psi}(t)\rangle \right) &= \hat{H}(t) \hat{U}(t)^{-1} |\tilde{\psi}(t)\rangle \\ i \frac{\partial}{\partial t} |\tilde{\psi}(t)\rangle &= \hat{U}(t) \hat{H}(t) \hat{U}(t)^{-1} |\tilde{\psi}(t)\rangle - i \hat{U}(t) \frac{\partial \hat{U}(t)^{-1}}{\partial t} |\tilde{\psi}(t)\rangle \\ i \frac{\partial}{\partial t} |\tilde{\psi}(t)\rangle &= \left(\hat{U}(t) \hat{H}(t) \hat{U}(t)^{-1} - i \hat{U}(t) \frac{\partial \hat{U}(t)^{-1}}{\partial t} \right) |\tilde{\psi}(t)\rangle. \end{aligned} \quad (2.2.4)$$

We have written the Schroedinger equation for the states $|\tilde{\psi}(t)\rangle$ in the rotating frame with the new Hamiltonian

$$\tilde{\hat{H}}(t) = \hat{U}(t) \hat{H}(t) \hat{U}(t)^{-1} - i \hat{U}(t) \frac{\partial \hat{U}(t)^{-1}}{\partial t}. \quad (2.2.5)$$

Using (2.2.5), it is now possible to define what we mean by writing the Hamiltonian in a superadiabatic basis. The key is to find the instantaneous eigenstates of the Hamiltonian in the original frame. Then we choose the unitary operator to be

$$\hat{U}(t) = \sum_i |n_i\rangle \langle n_i(t)|. \quad (2.2.6)$$

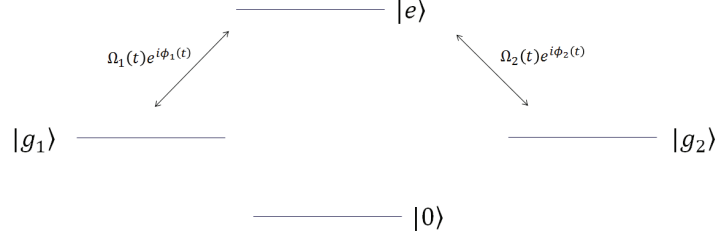


FIGURE 2.2.1. State-transfer diagram

Schematic representation of the state-transfer protocol described by the Hamiltonian of Eq. (2.2.9). The two ground states $|g_1\rangle$ and $|g_2\rangle$ are coupled to the excited state $|e\rangle$ via two time-dependent classical laser fields. The spectator state $|0\rangle$ does not couple to any other state (which is why we call it a spectator state).

Here $|n_i(t)\rangle$ correspond to the instantaneous eigenstates of the Hamiltonian at time t and $|n_i\rangle$ (often referred to as reference states) are simply the instantaneous eigenstates evaluated at time $t = 0$. By applying the unitary transformation (2.2.6) in (2.2.5), the transformed Hamiltonian will be given in the superadiabatic basis [30].

With these theoretical tools it is now possible to describe the state-transfer protocol that will be used throughout this thesis. The idea will be to consider a four-level system consisting of two ground states (which we will call $|g_1\rangle$ and $|g_2\rangle$), an excited state $|e\rangle$, and a spectator state $|0\rangle$. Initially, at time $t = 0$, we will consider a qubit state written as

$$|\psi(0)\rangle = \alpha |0\rangle + \beta |g_1\rangle. \quad (2.2.7)$$

The above state will represent our initial qubit state. The state-transfer protocol will consist of transferring this state into an intermediate qubit state, and then transferring the intermediate state back to the original state. There are a variety of reasons for doing this. For example, suppose the initial qubit state has a very short coherence time. We could transfer this quantum state to an intermediate state with a much longer coherence time to ensure that the state be preserved for longer periods of time. Another application would be to perform a phase gate. In this scenario, the goal is to transform the state $|\psi(0)\rangle$ as

$$|\psi(t_f)\rangle = \alpha |0\rangle + \beta e^{i\phi} |g_1\rangle. \quad (2.2.8)$$

Notice that in this case the state $|g_1\rangle$ picks up a phase. After writing down the Hamiltonian describing the couplings between the ground and excited states, we will show that this phase indeed corresponds to the geometric phase acquired by performing a cyclic evolution. In our case, the cyclic evolution will correspond to transferring our qubit state in (2.2.7) to an intermediate state and then back onto itself.

We now write down the Hamiltonian describing the couplings between the states mentioned above

$$\begin{aligned} \hat{H}(t) = & E_0 |0\rangle \langle 0| + \epsilon_{g_1} |g_1\rangle \langle g_1| + \epsilon_e |e\rangle \langle e| + \epsilon_{g_2} |g_2\rangle \langle g_2| \\ & + i \frac{\Omega_1(t)}{2} (|e\rangle \langle g_1| e^{-i\omega_1 t} - h.c.) + i \frac{\Omega_2(t)}{2} (e^{-i\phi(t)} |e\rangle \langle g_2| e^{-i\omega_2 t} - h.c.). \end{aligned} \quad (2.2.9)$$

The first line in (2.2.9) corresponds to the energy levels of the bare (undriven) states. The time-dependent functions $\Omega_1(t)$ and $\Omega_2(t)$ correspond to the classical laser field amplitudes which couple the ground states $|g_1\rangle$ and $|g_2\rangle$ to the excited state $|e\rangle$. We take $|0\rangle$ to be uncoupled and call this the spectator state. The time-dependent phase $\phi(t)$ represents the phase difference between our two drives. As will be shown below, it is very important that this phase be time-dependent to perform a phase gate. If this were not the case, the geometric phase would vanish.

We can greatly simplify the Hamiltonian of Eq. (2.2.9) if we choose to go into an interaction picture. By doing so, this will allow us to choose resonant frequencies for our lasers that cancel the frequencies ω_1 and ω_2 present in the coupling terms of the Hamiltonian. To see this, we define

$$\hat{H}_0 = E_0 |0\rangle \langle 0| + \epsilon_{g_1} |g_1\rangle \langle g_1| + \epsilon_e |e\rangle \langle e| + \epsilon_{g_2} |g_2\rangle \langle g_2|, \quad (2.2.10)$$

and

$$\hat{H}_1(t) = i \frac{\Omega_1(t)}{2} (|e\rangle \langle g_1| e^{-i\omega_1 t} - h.c.) + i \frac{\Omega_2(t)}{2} (e^{-i\phi(t)} |e\rangle \langle g_2| e^{-i\omega_2 t} - h.c.). \quad (2.2.11)$$

The interaction-picture Hamiltonian will then be given by

$$\hat{V}_I(t) = e^{i\hat{H}_0 t} \hat{H}_1(t) e^{-i\hat{H}_0 t}. \quad (2.2.12)$$

After performing this transformation, we obtain

$$\hat{V}_I(t) = i \frac{\Omega_1(t)}{2} \left(|e\rangle \langle g_1| e^{-i(\omega_1 + \omega_{g_1} - \omega_e)t} - h.c. \right) + i \frac{\Omega_2(t)}{2} \left(e^{-i\phi(t)} |e\rangle \langle g_2| e^{-i(\omega_2 + \omega_{g_2} - \omega_e)t} - h.c. \right). \quad (2.2.13)$$

It can immediately be seen from (2.2.13) that by choosing $\omega_1 = \omega_e - \omega_{g_1}$ and $\omega_2 = \omega_e - \omega_{g_2}$ (which is the required condition for resonance), we get a simple form for our interaction-picture Hamiltonian given by

$$\hat{V}_I(t) = i \frac{\Omega_1(t)}{2} (|e\rangle \langle g_1| - h.c.) + i \frac{\Omega_2(t)}{2} (e^{-i\phi(t)} |e\rangle \langle g_2| - h.c.). \quad (2.2.14)$$

As we showed in Eq. (2.1.3), the solution to the Schroedinger equation under an adiabatic approximation is given by a linear combination of the instantaneous eigenstates of the Hamiltonian with the appropriate phase factors (including both the dynamical and geometric phases). The next step is to find the eigenstates of the Hamiltonian (2.2.14). Out of the four eigenstates of this Hamiltonian, there will only be one (apart from the spectator state $|0\rangle$) which does not couple to the excited state. We will call this state the “dark state” represented by $|d(t)\rangle$. Diagonalizing (2.2.14) in the $\{|0\rangle, |g_1\rangle, |g_2\rangle, |e\rangle\}$ basis, the normalized eigenstates are given by

$$|0\rangle, \quad (2.2.15)$$

$$|d(t)\rangle = -\frac{\Omega_2(t)}{\sqrt{\Omega_1^2(t) + \Omega_2^2(t)}} |g_1\rangle + \frac{\Omega_1(t)e^{i\phi(t)}}{\sqrt{\Omega_1^2(t) + \Omega_2^2(t)}} |g_2\rangle, \quad (2.2.16)$$

$$|+(t)\rangle = \frac{1}{\sqrt{2}} \left(\frac{\Omega_1(t)}{\sqrt{\Omega_1^2(t) + \Omega_2^2(t)}} |g_1\rangle + i |e\rangle + \frac{\Omega_2(t)e^{i\phi(t)}}{\sqrt{\Omega_1^2(t) + \Omega_2^2(t)}} |g_2\rangle \right), \quad (2.2.17)$$

$$|-(t)\rangle = \frac{1}{\sqrt{2}} \left(\frac{\Omega_1(t)}{\sqrt{\Omega_1^2(t) + \Omega_2^2(t)}} |g_1\rangle - i |e\rangle + \frac{\Omega_2(t)e^{i\phi(t)}}{\sqrt{\Omega_1^2(t) + \Omega_2^2(t)}} |g_2\rangle \right). \quad (2.2.18)$$

The above eigenstates can be greatly simplified and written in a more intuitive way by performing a change of variables. We define the angle $\theta(t)$ and the parameter $G(t)$ by

$$\tan \theta(t) \equiv -\frac{\Omega_1(t)}{\Omega_2(t)}, \quad (2.2.19)$$

$$G(t) \equiv \frac{1}{2} \sqrt{\Omega_1^2(t) + \Omega_2^2(t)}. \quad (2.2.20)$$

With these definitions it is possible to rewrite the eigenstates as

$$|d(t)\rangle = \cos \theta(t) |g_1\rangle + e^{i\phi(t)} \sin \theta(t) |g_2\rangle, \quad (2.2.21)$$

$$|+(t)\rangle = \frac{1}{\sqrt{2}} \left(\sin \theta(t) |g_1\rangle + i |e\rangle - e^{i\phi(t)} \cos \theta(t) |g_2\rangle \right), \quad (2.2.22)$$

$$|-(t)\rangle = \frac{1}{\sqrt{2}} \left(\sin \theta(t) |g_1\rangle - i |e\rangle - e^{i\phi(t)} \cos \theta(t) |g_2\rangle \right). \quad (2.2.23)$$

The corresponding energies are

$$E_d = 0, \quad (2.2.24)$$

and

$$E_{\pm} = \pm G(t). \quad (2.2.25)$$

Notice that the states $|\pm(t)\rangle$ contain the excited state. For this reason these states will be called bright states [31]. In the interaction picture, the dark state has no dynamical phase since its energy eigenvalue vanishes. We now have all the ingredients to describe the closed-system quantum state-transfer protocol. Since $|d(t)\rangle$ has no amplitude to be in one of the excited states, then from (2.1.16) the transfer of information between any two qubit states can be generally described as

$$|\psi(t)\rangle = \alpha |0\rangle + \beta e^{i\gamma_d(t)} |d(t)\rangle. \quad (2.2.26)$$

Here, $\gamma_d(t)$ is the geometric phase associated with the dark state. Using (2.1.14) and (2.2.21), the geometric phase is given by

$$\gamma_d(t) = \int_0^t \dot{\phi}(t') \sin^2 \theta(t') dt'. \quad (2.2.27)$$

It can be seen that, if the laser phases were independent of time ($\dot{\phi} = 0$), then the geometric phase would vanish. Now, at the initial time, $t_i = 0$, we retrieve the qubit state of Eq. (2.2.7). At an intermediate time ($t_i < t_{int} < t_f$), $\theta(t_{int}) = (2n + 1) \frac{\pi}{2}$ with n an integer, (2.2.26) becomes

$$|\psi(t_{int})\rangle = \alpha |0\rangle + \beta e^{i\gamma_d(t_{int})} |g_2\rangle. \quad (2.2.28)$$

The above expression shows that when the initial qubit state ($\alpha |0\rangle + \beta |g_1\rangle$) is transferred to the second qubit state (2.2.28), the state $|g_2\rangle$ picks up a geometric phase (which is why at the beginning of this section we mentioned that this state transfer protocol would be well-suited for performing a phase gate). The integer $n \in \mathbb{N}$ describes the number of times the state is transferred (later it will be shown that performing the state-transfer protocol over many cycles will result in an improved fidelity). At the final time (t_f), the angle $\theta(t_f) = n\pi$ so that

$$|\psi(t_f)\rangle = \alpha |0\rangle + (-1)^n \beta e^{i\gamma_d(t_f)} |g_1\rangle. \quad (2.2.29)$$

Note that the relative sign difference between the spectator state ($|0\rangle$) and the first ground state ($|g_1\rangle$) is irrelevant since it can be absorbed into the phase $\gamma_d(t_f)$. In the next section we will see how to formulate the state-transfer protocol described above using a density-matrix approach. The main advantage to using density matrices will be clear when we consider open quantum systems. Since the off-diagonal components of the density matrix contain all the phase information (geometric and dynamical), it will be possible to obtain corrections to the geometric phase arising from the coupling to a quantum environment.

2.3. Density matrix approach

In this section we will show the general methods for obtaining the density matrix for closed-system dynamics. We will see that the geometric phase calculated in (2.2.27) will arise in computing the relevant off-diagonal element of the density matrix (also known as coherence).

For a pure state, the general form of the density matrix is given by

$$\hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)|. \quad (2.3.1)$$

Using (2.2.26), the density matrix for our system can be written as

$$\hat{\rho}(t) = |\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* e^{-i\gamma_d(t)} |0\rangle \langle d(t)| + \alpha^* \beta e^{i\gamma_d(t)} |d(t)\rangle \langle 0| + |\beta|^2 |d(t)\rangle \langle d(t)|. \quad (2.3.2)$$

Now, suppose we want to diagonalize our initial Hamiltonian. This diagonalization can be done by going into a rotating frame with the unitary operator

$$\hat{U}(t) = |0\rangle \langle 0| + |d\rangle \langle d(t)| + |+\rangle \langle +(t)| + |-\rangle \langle -(t)|. \quad (2.3.3)$$

Applying the operation $\tilde{H}(t) = \hat{U}(t) \hat{H}(t) \hat{U}^\dagger(t) - i\hat{U}(t) \dot{\hat{U}}^\dagger(t)$, the new Hamiltonian takes the form

$$\begin{aligned} \tilde{H}(t) = & G(t) \{ |+\rangle \langle +| - |-\rangle \langle -| \} + \dot{\phi} \sin^2 \theta(t) |d\rangle \langle d| \\ & + i \frac{\dot{\theta}}{\sqrt{2}} (|+\rangle \langle d| - |d\rangle \langle +| + |-\rangle \langle d| - |d\rangle \langle -|) - \frac{\dot{\phi}}{\sqrt{2}} (|+\rangle \langle d| + |d\rangle \langle +| + |-\rangle \langle d| + |d\rangle \langle -|) \\ & + \frac{\dot{\phi} \cos^2 \theta(t)}{2} (|+\rangle \langle +| + |-\rangle \langle +| + |+\rangle \langle -| + |-\rangle \langle -|). \end{aligned} \quad (2.3.4)$$

The above expression corresponds to the Hamiltonian written in the first superadiabatic eigenbasis. We say “first” superadiabatic eigenbasis because we could repeat this process to j^{th} order. If, for example, we were to find the eigenstates of the Hamiltonian in Eq. (2.3.4), we could write down a unitary operator of the same form as in (2.3.3) but written in terms of the instantaneous eigenstates of (2.3.4) instead of (2.2.14). We would perform the unitary transformation of (2.2.5) on (2.3.4) to find a new Hamiltonian written in the second-order adiabatic basis, and so on.

To obtain the density matrix in the rotating frame, one must apply the transformation $\tilde{\rho}(t) = \hat{U}(t) \hat{\rho}(t) \hat{U}^\dagger(t)$. Using (2.3.2) and (2.3.3) we find that

$$\tilde{\rho}(t) = |\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* e^{-i\gamma_d(t)} |0\rangle \langle d| + \alpha^* \beta e^{i\gamma_d(t)} |d\rangle \langle 0| + |\beta|^2 |d\rangle \langle d|. \quad (2.3.5)$$

Since only the dark ($|d\rangle$) and spectator ($|0\rangle$) states are present in our state transfer protocol, we will only be interested in the “ od ” component of the density matrix. In this case it will be useful to define

$$\tilde{\Pi}_{0d} \equiv |0\rangle \langle d| . \quad (2.3.6)$$

Throughout the entire thesis, whenever we take an average of an arbitrary operator \hat{A} , it will always be with respect to $\tilde{\rho}(t)$ in the following way

$$\langle \hat{A}(t) \rangle = \text{Tr} \left\{ \tilde{\rho}(t) \hat{A} \right\} \quad (2.3.7)$$

Using (2.3.5), the expectation value of the relevant component of the density matrix will be given by

$$\begin{aligned} \langle \tilde{\Pi}_{0d}(t) \rangle &= \text{Tr} \left\{ \tilde{\rho}(t) \tilde{\Pi}_{0d} \right\} = \text{Tr} \left\{ \tilde{\rho}(t) |0\rangle \langle d(t)| \right\} \\ &= \alpha\beta^* e^{-i\gamma_d(t)}. \end{aligned} \quad (2.3.8)$$

Note that $\tilde{\Pi}_{0d}(t)$ is in the Heisenberg picture with respect to the Hamiltonian $\tilde{H}(t)$.

At the time $t = 0$, (2.3.8) reduces to

$$\langle \tilde{\Pi}_{0d}(0) \rangle = \alpha\beta^* . \quad (2.3.9)$$

To find the “ $0d$ ” component of the density matrix in the lab frame we can obtain this term by “undoing” the unitary transformation we performed to get $\tilde{\rho}(t)$ or we can read it off of (2.3.2) so that

$$\hat{\Pi}_{0d}(t) = |0\rangle \langle d(t)| . \quad (2.3.10)$$

As we mentioned earlier, all the relevant phase information is contained in the “ $0d$ ” component of the density matrix. As expected, only the geometric phase is present since in the interaction picture the dark-state eigenenergy vanishes. If this were not the case, then one would expect that (2.3.8) also have a dynamical phase dependence.

When we consider open quantum systems, we can use the exact same procedure as used in this section to extract all the relevant phase information.

2.4. Review of cavity-photon shot noise

After developing the theoretical tools allowing us to compute corrections to the geometric phase for open quantum systems, we will apply our methods to adiabatic state transfer between qubits in a driven cavity. We will explicitly consider dephasing effects due to unavoidable photon shot noise. Consequently, it is necessary to review the physics behind cavity-photon shot noise. Much of what follows will be based on [40].

We consider a cavity driven by an external field. For a cavity to be driven, it is necessary to open one of its ports. Consequently, this will enable the cavity to leak energy in the surrounding bath. For high-Q cavities it is possible to make a distinction between the internal cavity modes and the external bath modes. We can write the Hamiltonian as

$$\hat{H} = \hat{H}_{sys} + \hat{H}_{bath} + \hat{H}_{int}. \quad (2.4.1)$$

The bath Hamiltonian will be described by a collection of harmonic modes

$$\hat{H}_{bath} = \sum_q \hbar \omega_q \hat{b}_q^\dagger \hat{b}_q. \quad (2.4.2)$$

The bath modes obey the commutation relations

$$[\hat{b}_q, \hat{b}_{q'}^\dagger] = \delta_{q,q'}. \quad (2.4.3)$$

Within a rotating-wave approximation, the coupling Hamiltonian is described as

$$\hat{H}_{int} = -i\hbar \sum_q \left[f_q \hat{a}^\dagger \hat{b}_q - f_q^* \hat{b}_q^\dagger \hat{a} \right]. \quad (2.4.4)$$

We neglected terms such as $\hat{a} \hat{b}_q$ and $\hat{a}^\dagger \hat{b}_q^\dagger$ since in a rotating-wave approximation they oscillate at high frequency and so have little effects on the dynamics. The cavity will be specified by a single degree of freedom obeying the bosonic commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (2.4.5)$$

The Heisenberg equation of motion for the bosonic bath modes is given by

$$\dot{\hat{b}}_q = -i\omega_q \hat{b}_q + f_q^* \hat{a}. \quad (2.4.6)$$

The second term on the right-hand side of (2.4.6) corresponds to a forcing term due to the motion of the cavity degree-of-freedom. Considering $t_0 < t$ to be a time in the past before a wave packet launched at the cavity has reached it, we can solve (2.4.6) exactly to obtain

$$\hat{b}_q(t) = e^{-i\omega_q(t-t_0)} \hat{b}_q(t_0) + \int_{t_0}^t d\tau e^{-i\omega_q(t-\tau)} f_q^* \hat{a}(\tau). \quad (2.4.7)$$

The second term on the right corresponds to a wave radiated by the cavity into the bath. If we had considered $t_1 > t$ to be a time in the future after the input field had interacted with the cavity, then the solution to (2.4.6) would instead take the form

$$\hat{b}_q(t) = e^{-i\omega_q(t-t_1)} \hat{b}_q(t_1) - \int_t^{t_1} d\tau e^{-i\omega_q(t-\tau)} f_q^* \hat{a}(\tau). \quad (2.4.8)$$

Notice that there is a sign difference in the second term compared to (2.4.7) arising from the fact that $t_1 > t$. Now we can write down the equation-of-motion for the cavity degree of freedom

$$\dot{\hat{a}} = \frac{i}{\hbar} [\hat{H}_{sys}, \hat{a}] - \sum_q f_q \hat{b}_q. \quad (2.4.9)$$

It is important to note that so far we have not specified the dynamics of the cavity, so that the first term in (2.4.9) is left completely general. We can use (2.4.7) in the second term of (2.4.9) to get

$$\begin{aligned} \sum_q f_q \hat{b}_q &= \sum_q f_q e^{-i\omega_q(t-t_0)} \hat{b}_q(t_0) \\ &\quad + \sum_q |f_q|^2 \int_{t_0}^t d\tau e^{-i(\omega_q - \omega_c)(t-\tau)} \left[e^{i\omega_c(t-\tau)} \hat{a}(\tau) \right]. \end{aligned} \quad (2.4.10)$$

We can simplify the last result by noting that if we considered the cavity to be a simple harmonic mode of frequency ω_c , then we could represent the decay rate from the $n = 1$

single-photon excited state to the $n = 0$ ground state by a Fermi golden rule expression

$$\kappa(\omega_c) = 2\pi \sum_q |f_q|^2 \delta(\omega_c - \omega_q). \quad (2.4.11)$$

Fourier transforming, we find

$$\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \kappa(\omega_c + \nu) e^{-i\nu(t-\tau)} = \sum_q |f_q|^2 e^{-i(\omega_q - \omega_c)(t-\tau)}. \quad (2.4.12)$$

In a Markov approximation, we set $\kappa(\nu) = \kappa$ to be a constant over the cavity frequencies.

Using the fact that $\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} e^{-i\nu(t-\tau)} = \delta(t - \tau)$, (2.4.12) can be simplified to

$$\sum_q |f_q|^2 e^{-i(\omega_q - \omega_c)(t-\tau)} = \kappa \delta(t - \tau). \quad (2.4.13)$$

Within the range of validity of the Markov approximation, we can also set $t_0 = -\infty$. Using $\int_{-\infty}^{\tau} \delta(t' - \tau) dt' = \frac{1}{2}$, the equation-of-motion for the cavity degree of freedom simplifies to

$$\dot{\hat{a}} = \frac{i}{\hbar} [\hat{H}_{sys}, \hat{a}] - \frac{\kappa}{2} \hat{a} - \sum_q f_q e^{-i\omega_q(t-t_0)} \hat{b}_q(t_0). \quad (2.4.14)$$

The term representing the wave radiated by the cavity is now a simple linear damping term under a Markov approximation. The factor of 2 indicates that the amplitude decays at half the rate of the intensity. By performing a Markov approximation, we can approximate $f \equiv \sqrt{|f_q|^2}$ to be a constant and also set the density of states $\rho = \sum_q \delta(\omega_c - \omega_q)$ to be a constant [40]. Then from (2.4.11) we get the very simple result

$$\kappa = 2\pi f^2 \rho. \quad (2.4.15)$$

At this stage it will prove convenient to define the “input mode” by

$$\hat{b}_{in}(t) \equiv \frac{1}{\sqrt{2\pi\rho}} \sum_q e^{-i\omega_q(t-t_0)} \hat{b}_q(t_0). \quad (2.4.16)$$

Then with this definition, we can rewrite the equation-of-motion for the cavity mode to be

$$\dot{\hat{a}} = \frac{i}{\hbar} [\hat{H}_{sys}, \hat{a}] - \frac{\kappa}{2} \hat{a} - \sqrt{\kappa} \hat{b}_{in}(t). \quad (2.4.17)$$

The input mode will evolve freely until it comes into contact with the cavity at which point it will begin driving the cavity. It is sound to interpret $\hat{b}_{in}(t)$ as an input mode since it evolves under the free bath Hamiltonian and acts as a driving term in the equation-of-motion for the cavity mode. Following along the same reasoning as described above, we can also define an output mode from Eq. (2.4.8) by

$$\hat{b}_{out}(t) \equiv \frac{1}{\sqrt{2\pi\rho}} \sum_q e^{-i\omega_q(t-t_1)} \hat{b}_q(t_1). \quad (2.4.18)$$

We can interpret this as an output mode since it is simply the free evolution of the bath modes in the distant future after interacting with the cavity. Note that it is also possible to write an equation-of-motion for the cavity mode in terms of the output field. This is given by

$$\dot{\hat{a}} = \frac{i}{\hbar} [\hat{H}_{sys}, \hat{a}] + \frac{\kappa}{2} \hat{a} - \sqrt{\kappa} \hat{b}_{out}(t). \quad (2.4.19)$$

If we subtract (2.4.19) from (2.4.17), then we can write the output field in terms of the input field as

$$\hat{b}_{out}(t) = \hat{b}_{in}(t) + \sqrt{\kappa} \hat{a}(t). \quad (2.4.20)$$

This is consistent with the view that the output field should be a sum of a reflected incoming field plus the field radiated by the cavity.

So far we have considered a completely general Hamiltonian for the cavity dynamics (apart from being restricted to a single mode). We can now consider the specific case where the cavity is comprised of a single harmonic oscillator with frequency ω_c . In this case the system Hamiltonian would be given by

$$\hat{H}_{sys} = \hbar\omega_c \hat{a}^\dagger \hat{a}. \quad (2.4.21)$$

The commutator in (2.4.17) is now straightforward to compute and so the cavity equation-of-motion reduces to

$$\dot{\hat{a}} = -i\omega_c \hat{a} - \frac{\kappa}{2} \hat{a} - \sqrt{\kappa} \hat{b}_{in}(t). \quad (2.4.22)$$

A convenient trick for solving this equation is to first perform a Fourier transform to write a solution for $\hat{a}[\omega] = \int_{-\infty}^{\infty} dt \hat{a}(t) e^{i\omega t}$. Proceeding in this manner we find

$$\hat{a}[\omega] = -\sqrt{\kappa} \chi_c [\omega - \omega_c] \hat{b}_{in}[\omega]. \quad (2.4.23)$$

Here we defined the susceptibility of the cavity by

$$\chi_c [\omega - \omega_c] \equiv \frac{1}{-i(\omega - \omega_c) + \kappa/2}. \quad (2.4.24)$$

We can also solve for the output field in terms of the input field which will be given by

$$\hat{b}_{out}[\omega] = \frac{\omega - \omega_c - i\kappa/2}{\omega - \omega_c + i\kappa/2} \hat{b}_{in}[\omega]. \quad (2.4.25)$$

If we drive the cavity on resonance so that $\omega = \omega_c$, the output field assumes the simple form

$$\hat{b}_{out}[\omega] = \frac{\sqrt{\kappa}}{2} \hat{a}[\omega]. \quad (2.4.26)$$

The equation-of-motion for the cavity mode can also be solved in the time domain. The solution is given by

$$\hat{a}(t) = e^{-(i\omega_c + \kappa/2)(t-t_0)} \hat{a}(t_0) - \sqrt{\kappa} \int_{t_0}^t d\tau e^{-(i\omega_c + \kappa/2)(t-\tau)} \hat{b}_{in}(\tau). \quad (2.4.27)$$

We can specialize to the case where the input field is a coherent drive at a frequency $\omega_L = \omega_c + \Delta$ with a classical and quantum amplitude given by

$$\hat{b}_{in}(t) = e^{-i\omega_L t} \left[\bar{b}_{in} + \hat{\xi}(t) \right]. \quad (2.4.28)$$

Here \bar{b}_{in} is the classical amplitude and $\hat{\xi}(t)$ is the quantum amplitude. We can also take $t_0 \rightarrow \infty$ in (2.4.27) which corresponds to having the initial transient in the cavity damped out. In this case the solution to (2.4.27) will be given by

$$\hat{a}(t) = e^{-i\omega_L t} \left[\bar{a} + \hat{d}(t) \right], \quad (2.4.29)$$

with the classical contribution \bar{a} given by

$$\bar{a} = -\frac{\sqrt{\kappa}}{-i\Delta + \kappa/2} \bar{b}_{in}. \quad (2.4.30)$$

where Δ is the detuning frequency. In the frame rotating at the drive frequency, the quantum contribution will be given by

$$\hat{d}(t) = -\sqrt{\kappa} \int_{-\infty}^t d\tau e^{(i\Delta - \kappa/2)(t-\tau)} \hat{\xi}(\tau). \quad (2.4.31)$$

Using (2.4.16) and (2.4.28), we can obtain the commutation relations for the fields $\hat{\xi}(t)$

$$\begin{aligned} [\hat{b}_{in}(t), \hat{b}_{in}^\dagger(t')] &= [\hat{\xi}(t), \hat{\xi}^\dagger(t')] \\ &= \frac{1}{2\pi\rho} \sum_q e^{-i(\omega_q - \omega_L)(t-t')} \\ &= \delta(t - t'). \end{aligned} \quad (2.4.32)$$

We can also verify that indeed the correct commutation relations for the time-dependent cavity modes are satisfied by using (2.4.29), (2.4.31) and (2.4.32)

$$\begin{aligned} [\hat{a}(t), \hat{a}^\dagger(t)] &= [\hat{d}(t), \hat{d}^\dagger(t)] \\ &= \kappa \int_{-\infty}^t d\tau \int_{-\infty}^t d\tau' e^{-(-i\Delta + \kappa/2)(t-\tau)} e^{-(i\Delta + \kappa/2)(t-\tau')} \delta(\tau - \tau') \\ &= 1. \end{aligned} \quad (2.4.33)$$

Since the port of the cavity is open, vacuum noise will enter the cavity creating zero-point fluctuations [40].

It is now possible to give an explanation for the quantum noise in the number of photons inside the cavity. Based on the picture that we have developed, this noise will be due to the vacuum noise that enters through the cavity port that was brought in by the classical field. There will be interference between the vacuum noise and the classical drive that will lead to fluctuations in the number of photons inside the cavity.

As a last remark, we should also think about temperature. The field $\hat{\xi}$ will contain thermal radiation when in thermal equilibrium. Recall that when making the Markov approximation, we assumed that the bath was being probed over a very broad range of frequencies centered on ω_c . To a good approximation [40], we have

$$\left\langle \hat{\xi}^\dagger(t) \hat{\xi}(t') \right\rangle = N_{th} \delta(t - t'), \quad (2.4.34)$$

$$\left\langle \hat{\xi}(t) \hat{\xi}^\dagger(t') \right\rangle = (N_{th} + 1) \delta(t - t'), \quad (2.4.35)$$

where $N_{th} = n_B(\hbar\omega_c)$ is the bosonic thermal equilibrium occupation number of the mode at the frequency of interest. The relations (2.4.34) and (2.4.35) will be used extensively in chapter 4 when we apply our state transfer methods to cavity-photon shot noise. As a final note, using (2.4.31), (2.4.34) and (2.4.35), we find that the correlation function of the \hat{d} operator is given by

$$\left\langle \hat{d}^\dagger(t) \hat{d}(t') \right\rangle = N_{th} e^{i\Delta(t-t')} e^{-\frac{\kappa}{2}|t-t'|}. \quad (2.4.36)$$

Open-system evolution under the secular approximation

3.1. Density matrix phase

In this section we will develop the general theoretical tools that are required to calculate dephasing effects for the case where our four-level system is coupled to a quantum dissipative bath. The starting point will be to consider the Hamiltonian that we obtained in (2.2.14) and add the noisy terms $\delta\hat{\omega}_1 |g_1\rangle\langle g_1| + \delta\hat{\omega}_2 |g_2\rangle\langle g_2|$. The functions $\delta\hat{\omega}_1$ and $\delta\hat{\omega}_2$ correspond to bath degrees-of-freedom which couple to the ground states of our four-level system. The interaction-picture Hamiltonian will now be given by

$$\begin{aligned} \hat{H}(t) = & i\frac{\Omega_1(t)}{2} (|e\rangle\langle g_1| - |g_1\rangle\langle e|) + i\frac{\Omega_2(t)}{2} (e^{-i\phi(t)} |e\rangle\langle g_2| - e^{i\phi(t)} |g_2\rangle\langle e|) \\ & + \delta\hat{\omega}_1 |g_1\rangle\langle g_1| + \delta\hat{\omega}_2 |g_2\rangle\langle g_2| + \hat{H}_{env}(\delta\hat{\omega}_1, \delta\hat{\omega}_2). \end{aligned} \quad (3.1.1)$$

The term $\hat{H}_{env}(\delta\hat{\omega}_1, \delta\hat{\omega}_2)$ in (3.1.1) dictates the dynamics of the bath degrees of freedom. For now we will consider this to be completely general since its specific operator dependence will not be relevant.

The first step is to go into a rotating frame by choosing a unitary transformation \hat{U} which diagonalizes $\hat{H}_0(t)$ where

$$\hat{H}_0(t) = i\frac{\Omega_1(t)}{2} (|e\rangle\langle g_1| - |g_1\rangle\langle e|) + i\frac{\Omega_2(t)}{2} (e^{-i\phi(t)} |e\rangle\langle g_2| - e^{i\phi(t)} |g_2\rangle\langle e|). \quad (3.1.2)$$

In order to do so we choose the unitary transformation of (2.3.3). The states $|d\rangle$, $|+\rangle$ and $|-\rangle$ are reference states chosen to be the dark and bright states evaluated at time $t = 0$. Notice that the situation is different than what we were dealing with in chapter 2. For a purely closed system, we chose a unitary transformation that diagonalized the entire Hamiltonian.

For the open-system case, we don't consider a unitary transformation given in terms of the eigenstates of the entire Hamiltonian. Instead, we only choose the eigenstates of $\hat{H}_0(t)$. The transformed Hamiltonian will take the form $\tilde{H}(t) = \hat{U} \hat{H}(t) \hat{U}^\dagger - i\hat{U} \dot{\hat{U}}^\dagger$. The result of this transformation is given by

$$\begin{aligned}
\tilde{H}(t) = & G(t) \{ |+\rangle \langle +| - |-\rangle \langle -| \} + (\dot{\phi} \sin^2 \theta(t) + \delta\hat{\omega}_1 \cos^2 \theta(t) + \delta\hat{\omega}_2 \sin^2 \theta(t)) |d\rangle \langle d| \\
& + (\delta\hat{\omega}_1 - \delta\hat{\omega}_2) \frac{\cos \theta(t) \sin \theta(t)}{\sqrt{2}} (|+\rangle \langle d| + |-\rangle \langle d| + |d\rangle \langle +| + |d\rangle \langle -|) \\
& + \frac{(\delta\hat{\omega}_1 \sin^2 \theta(t) + \delta\hat{\omega}_2 \cos^2 \theta(t))}{2} (|+\rangle \langle +| + |-\rangle \langle +| + |+\rangle \langle -| + |-\rangle \langle -|) \\
& + i \frac{\dot{\theta}}{\sqrt{2}} (|+\rangle \langle d| - |d\rangle \langle +| + |-\rangle \langle d| - |d\rangle \langle -|) - \frac{\dot{\phi}}{\sqrt{2}} (|+\rangle \langle d| + |d\rangle \langle +| + |-\rangle \langle d| + |d\rangle \langle -|) \\
& + \frac{\dot{\phi} \cos^2 \theta(t)}{2} (|+\rangle \langle +| + |-\rangle \langle +| + |+\rangle \langle -| + |-\rangle \langle -|) + \hat{H}_{env}(\delta\hat{\omega}_1, \delta\hat{\omega}_2). \quad (3.1.3)
\end{aligned}$$

At first it might seem hopeless to make any progress with this Hamiltonian without resorting to approximation methods such as Bloch-Redfield theory or other similar means which would limit our results to Markovian environments. However, the Hamiltonian can be greatly simplified by performing a secular approximation. All the off-diagonal terms of the Hamiltonian will enter in perturbation theory with suppression factors $\sim \frac{1}{G}$ compared to the diagonal terms. If we consider the limit where $\frac{\dot{\theta}}{G} \ll 1$, $\frac{\dot{\phi}}{G} \ll 1$ and $\frac{|\delta\hat{\omega}_i|}{G} \ll 1$ (which can be achieved by considering very large laser amplitudes relative to the other parameters of the system), then we can drop all the off-diagonal terms of the Hamiltonian. This defines the secular approximation. Hence we get

$$\tilde{H}_{sec}(t) = \hat{V}_{sec}(t) + \hat{H}_{env}(\delta\hat{\omega}_1, \delta\hat{\omega}_2), \quad (3.1.4)$$

where

$$\begin{aligned}
\hat{V}_{sec}(t) \equiv & G(t) \{ |+\rangle \langle +| - |-\rangle \langle -| \} + (\dot{\phi} \sin^2 \theta(t) + \delta\hat{\omega}_1 \cos^2 \theta(t) + \delta\hat{\omega}_2 \sin^2 \theta(t)) |d\rangle \langle d| \\
& + \frac{(\dot{\phi} \cos^2 \theta(t) + \delta\hat{\omega}_1 \sin^2 \theta(t) + \delta\hat{\omega}_2 \cos^2 \theta(t))}{2} (|+\rangle \langle +| + |-\rangle \langle -|) \quad (3.1.5)
\end{aligned}$$

This Hamiltonian is clearly much simpler than the previous one. As will be shown below, the great technical advantage in performing the secular approximation is that we won't need to solve a master equation using Bloch-Redfield theory to obtain the phase information of the density matrix. Furthermore, Bloch-Redfield theory would only be valid for systems weakly coupled to an environment with a short correlation time (i.e. a Markovian environment) relative to the decay time of the off-diagonal component of the density matrix [41]. In our approach, the phase information will be obtained from the equation-of-motion for the off-diagonal component of the density matrix. Since the Hamiltonian written in the superadiabatic basis is purely diagonal after performing a secular approximation, its commutation relations with the component of interest of the density matrix will turn out to be fairly simple.

We start by calculating the average of $\langle \tilde{\Pi}_{0d}(t) \rangle$. After obtaining the thermal average of $\langle \tilde{\Pi}_{0d}(t) \rangle$ in the rotating frame (see (3.1.6) written below), it will be a simple matter of undoing the unitary transformations following (2.3.10) to obtain the desired component in the lab frame. For a system-bath coupling, the previous average can be written as

$$\langle \tilde{\Pi}_{0d}(t) \rangle = \text{Tr} \left\{ \tilde{\Pi}_{0d}(t) \tilde{\rho}(0) \right\}, \quad (3.1.6)$$

where $\tilde{\rho}(0) = \tilde{\rho}_S(0) \otimes \hat{\rho}_{env}$ with $\hat{\rho}_{env} = \frac{1}{Z_{env}} e^{-\beta \hat{H}_{env}}$ and $Z_{env} = \text{Tr} \left\{ e^{-\beta \hat{H}_{env}} \right\}$ (note that we can shift the time dependence between $\tilde{\Pi}_{0d}$ and $\tilde{\rho}$ by using the cyclic permutation properties of the trace). This assumes that at the initial time, the system and bath are uncorrelated and that the environment is in thermal equilibrium. Later in chapter 4, we will consider cases where the environment is not in thermal equilibrium and so $\hat{\rho}_{env}$ will have to be modified. The term $\tilde{\rho}_S(0)$ corresponds to the density matrix of our system of interest.

To make further progress, we write down the equation-of-motion for $\tilde{\Pi}_{0d}(t)$ in the interaction picture with respect to $\hat{H}_{env}(\delta\hat{\omega}_1, \delta\hat{\omega}_2)$. The trace in (3.1.6) can be written as

$$\langle \tilde{\Pi}_{0d}(t) \rangle = \text{Tr} \left\{ \tilde{\Pi}_{I'0d}(t) \tilde{\rho}(0) \right\}, \quad (3.1.7)$$

where the subscript I' denotes an operator that is in the interaction picture with respect to $\hat{H}_{env}(\delta\hat{\omega}_1, \delta\hat{\omega}_2)$. The equation-of-motion is then given by

$$\frac{d}{dt}\tilde{\Pi}_{I'0d}(t) = i \left[\hat{V}_{secI'}(t), \tilde{\Pi}_{I'0d}(t) \right]. \quad (3.1.8)$$

Notice that in (3.1.8), $\tilde{\Pi}_{I'0d}(t)$ has explicit time dependence. One possible trick for getting rid of this time dependence is to integrate the equation of motion using an iterative procedure. Integrating the equation-of-motion we get

$$\tilde{\Pi}_{I'0d}(t) = \tilde{\Pi}_{I'0d}(0) + i \int_0^t dt' \left[\hat{V}_{secI'}(t'), \tilde{\Pi}_{I'0d}(t') \right]. \quad (3.1.9)$$

We can iterate this result by inserting the first term as a zeroth-order solution and repeating this process for all higher orders. Doing this we find

$$\tilde{\Pi}_{I'0d}(t) = \tilde{\Pi}_{I'0d}(0) + i \int_0^t dt_1 \left[\hat{V}_{secI'}(t_1), \tilde{\Pi}_{I'0d}(0) \right] + (i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \left[\hat{V}_{secI'}(t_1), \left[\hat{V}_{secI'}(t_2), \tilde{\Pi}_{I'0d}(0) \right] \right] + \dots \quad (3.1.10)$$

As can be seen from (3.1.10), $\tilde{\Pi}_{I'0d}$ no longer depends on time and it is a straightforward matter to perform the commutation relation using (2.3.6). First, however, we can rewrite (3.1.10) in a much simpler form as

$$\tilde{\Pi}_{I'0d}(t) = T_t e^{i \int_0^t dt' L_1(t')} \tilde{\Pi}_{I'0d}(0), \quad (3.1.11)$$

where T_t is the time-ordering operator. By time-ordering operator, we mean that all operators evaluated at later times appear to the left of those evaluated at earlier times. So as an example, for a product of two operators we could write

$$T_t \hat{A}(t_1) \hat{B}(t_2) = \hat{A}(t_1) \hat{B}(t_2) \Theta(t_1 - t_2) \pm \hat{B}(t_2) \hat{A}(t_1) \Theta(t_2 - t_1). \quad (3.1.12)$$

The superoperator $L_1(t)$ acts in the following way

$$L_1(t) \tilde{\Pi}_{I'0d}(0) = \left[\hat{V}_{secI'}(t), \tilde{\Pi}_{I'0d}(0) \right]. \quad (3.1.13)$$

Using the fact that $[|d\rangle\langle d|, |0\rangle\langle d|] = -|0\rangle\langle d|$, the commutator is evaluated to be

$$\left[\hat{V}_{secI'}(t), \tilde{\Pi}_{I'0d}(0)\right] = -\left\{\dot{\phi}\sin^2\theta(t) + \delta\hat{\omega}_1(t)\cos^2\theta(t) + \delta\hat{\omega}_2(t)\sin^2\theta(t)\right\}|0\rangle\langle d|, \quad (3.1.14)$$

where $\delta\hat{\omega}_1(t)$ and $\delta\hat{\omega}_2(t)$ are in the interaction picture with respect to $\hat{H}_{env}(\delta\hat{\omega}_1, \delta\hat{\omega}_2)$.

Consequently, we find that

$$\tilde{\Pi}_{I'0d}(t) = T_t e^{-i\int_0^t dt' (\dot{\phi}\sin^2\theta(t') + \delta\hat{\omega}_1(t')\cos^2\theta(t') + \delta\hat{\omega}_2(t')\sin^2\theta(t'))} \tilde{\Pi}_{0d}, \quad (3.1.15)$$

where $\tilde{\Pi}_{0d}$ was defined in (2.3.6). Inserting this result into the trace of Eq. (3.1.7), we find that

$$\left\langle\tilde{\Pi}_{0d}(t)\right\rangle = \text{Tr}\left\{T_t e^{-i\int_0^t dt' (\dot{\phi}\sin^2\theta(t') + \delta\hat{\omega}_1(t')\cos^2\theta(t') + \delta\hat{\omega}_2(t')\sin^2\theta(t'))} \tilde{\Pi}_{0d} \tilde{\rho}_S(0) \otimes \hat{\rho}_{env}\right\}. \quad (3.1.16)$$

Using the identity $\text{Tr}\{\hat{A} \otimes \hat{B}\} = \text{Tr}\{\hat{A}\} \text{Tr}\{\hat{B}\}$ equation (3.1.16) reduces to

$$\left\langle\tilde{\Pi}_{0d}(t)\right\rangle = \left\langle\tilde{\Pi}_{0d}(0)\right\rangle \left\langle T_t e^{-i\int_0^t dt' (\dot{\phi}\sin^2\theta(t') + \delta\hat{\omega}_1(t')\cos^2\theta(t') + \delta\hat{\omega}_2(t')\sin^2\theta(t'))} \right\rangle. \quad (3.1.17)$$

Since the average in the second term of equation (3.1.17) is with respect to the environmental bath modes, we can simplify our expression by writing

$$\left\langle\tilde{\Pi}_{0d}(t)\right\rangle = e^{-i\gamma_d(t)} \left\langle\tilde{\Pi}_{0d}(0)\right\rangle \left\langle T_t e^{-i\int_0^t dt' (\delta\hat{\omega}_1(t')\cos^2\theta(t') + \delta\hat{\omega}_2(t')\sin^2\theta(t'))} \right\rangle. \quad (3.1.18)$$

Notice that the correction to the geometric phase arising from the system-bath coupling is now transparent. The term inside the thermal average corresponds to corrections to the geometric phase which will give rise to dephasing (which is why these types of contributions are called geometric dephasing [29]). We also mention that the average $\left\langle\tilde{\Pi}_{0d}(0)\right\rangle$ is given by (2.3.9).

3.2. Antisymmetric noise $\delta\hat{\omega}_1 = -\delta\hat{\omega}_2$

In this section we consider the case of purely antisymmetric noise so that $\delta\hat{\omega}_1 = -\delta\hat{\omega}_2$.

In this case (3.1.18) will simplify to

$$\left\langle \tilde{\Pi}_{0d}(t) \right\rangle = \left\langle \tilde{\Pi}_{0d}(0) \right\rangle e^{-i\gamma_d(t)} \left\langle T_t e^{-i \int_0^t dt' \cos(2\theta(t')) \delta\hat{\omega}_1(t')} \right\rangle. \quad (3.2.1)$$

This is a crucial step for later calculations to come. By picking $\delta\hat{\omega}_1 = -\delta\hat{\omega}_2$, we are considering a situation where there are strong correlations between the noise hitting the two levels $|g_1\rangle$ and $|g_2\rangle$. This opens up a very interesting possibility, the idea of using this correlation to suppress decoherence while one is still doing the desired adiabatic evolution protocol. We refer the reader to chapter 4 for more details. Note that in chapter 4, it will be possible to choose reasonable parameters so that the cavity shot noise will be in the regime satisfying $\delta\hat{\omega}_1 = -\delta\hat{\omega}_2$.

It is also important to understand the origin of the function $\cos(2\theta(t))$ appearing in (3.2.1). If $\theta(t) = 0$ or $\theta(t) = \frac{\pi}{2}$, it is clear from (2.2.21) and (2.2.26) that the state is either all $|g_1\rangle$ or all $|g_2\rangle$. In this case, we would only see the $\delta\hat{\omega}_1$ or $\delta\hat{\omega}_2$ noise and hence the correlations aren't important. Therefore, we would expect the noise to be maximal for $\theta(t) = 0$ or $\theta(t) = \frac{\pi}{2}$. On the other hand, for $\theta(t) = \frac{\pi}{4}$, the state (2.2.26) is in an equal superposition of $|g_1\rangle$ and $|g_2\rangle$. In this case the average of the noise Hamiltonian is $\delta\hat{\omega}_1 + \delta\hat{\omega}_2 = 0$. Thus the state (2.2.26) sees both the $\delta\hat{\omega}_1$ and $\delta\hat{\omega}_2$ noises equally; as they are perfectly anti-correlated, the net contribution is always zero. All these arguments are quantitatively understood from the function $\cos(2\theta(t))$.

The quantity that we will mostly be interested in when performing our state-transfer protocols is the fidelity. This quantity, which takes on values between 0 and 1, will describe how “close” our final state is to the original one. So for a perfect state transfer with no accumulated phase difference between the spectator and ground state, the fidelity would be unity. Formally, the fidelity is defined as

$$F \equiv \text{Tr} \left\{ \tilde{\rho}(t_f) |\psi_s(t_i)\rangle \langle \psi_s(t_i)| \right\}. \quad (3.2.2)$$

The state $|\psi_s(t_i)\rangle$ is given by (2.2.7) and we used the fact that at the initial time, the density matrix is described by a pure state and can thus be written as

$$\tilde{\rho}_s(t_i) = |\psi_s(t_i)\rangle \langle \psi_s(t_i)| \quad (3.2.3)$$

The fidelity can be written in terms of the relevant component of the density matrix using the following procedure. In general, the average $\langle \tilde{\Pi}_{\alpha\beta}(t_i) \rangle$ is given by

$$\langle \tilde{\Pi}_{\alpha\beta}(t) \rangle \equiv \text{Tr} \{ |\alpha\rangle \langle \beta| \tilde{\rho}(t) \}. \quad (3.2.4)$$

Using the relation (2.3.5) and (3.2.4), the fidelity reduces to

$$F = |\alpha|^2 \langle \tilde{\Pi}_{00}(t_f) \rangle + 2\text{Re} \left[\alpha^* \beta \langle \tilde{\Pi}_{0d}(t_f) \rangle \right] + |\beta|^2 \langle \tilde{\Pi}_{dd}(t_f) \rangle, \quad (3.2.5)$$

where we used (2.3.5) to obtain $\langle \tilde{\Pi}_{00}(t_i) \rangle = |\alpha|^2$ and $\langle \tilde{\Pi}_{dd}(t_i) \rangle = |\beta|^2$. Since $\left[|0\rangle \langle 0|, \hat{V}_{secI'}(t) \right] = \left[|d\rangle \langle d|, \hat{V}_{secI'}(t) \right] = 0$, then we can follow the same logic that was used to go from (3.1.6) to (3.1.17). In this case we find that

$$\langle \tilde{\Pi}_{00}(t_f) \rangle = |\alpha|^2, \quad (3.2.6)$$

and

$$\langle \tilde{\Pi}_{dd}(t_f) \rangle = |\beta|^2. \quad (3.2.7)$$

The fidelity then reduces to

$$F = |\alpha|^4 + |\beta|^4 + 2\text{Re} \left[\alpha^* \beta \langle \tilde{\Pi}_{0d}(t_f) \rangle \right]. \quad (3.2.8)$$

We can use (3.2.1) to write the fidelity as

$$F = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \text{Re} \left[e^{-i\gamma_d(t_f)} \left\langle T_t e^{-i \int_0^{t_f} dt' \cos(2\theta(t')) \delta\hat{\omega}_1(t')} \right\rangle \right]. \quad (3.2.9)$$

This is one of the central results of our work and will be used throughout this thesis. If the system were not coupled to a dissipative quantum environment, then the fidelity would simply be given by

$$F = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \cos \gamma_d(t_f) \quad (3.2.10)$$

When the system is coupled to a quantum dissipative environment, the fidelity will be given by (3.2.9) which has an extra contribution which will depend on the spectral density of the bath. The reason is that we must perform an average over the bath degrees of freedom

and so will be faced with calculating correlation functions of the form $\langle \delta\hat{\omega}_1(t_1) \delta\hat{\omega}_1(t_2) \rangle$. Consequently, the dynamics of the bath will determine how the geometric phase will be modified due to the quantum system-bath coupling. Recall that these results are valid in the large- G limit. If this were not the case, we would need to include all the off-diagonal components of the superadiabatic Hamiltonian which would then give extra contributions to the geometric phase which are not accounted for in (3.2.9).

We conclude this section with a neat result that stems from taking a particular spectral density for the bath degrees-of-freedom. To simplify the notation, we define

$$\hat{X}(t) \equiv \int_0^t dt' \cos(2\theta(t')) \delta\hat{\omega}_1(t'). \quad (3.2.11)$$

When performing a statistical ensemble average over the realizations of the bath degrees-of-freedom, we assume that the central-limit theorem applies so that

$$\langle e^{-i\hat{X}(t)} \rangle \approx e^{-\frac{1}{2}\langle \hat{X}^2(t) \rangle}. \quad (3.2.12)$$

In getting the above result, we effectively treat $\hat{X}(t)$ as Gaussian, which implies that t in (3.2.12) is in general much longer than the correlation length of the noise coming from $\delta\hat{\omega}_1(t)$ (or that $\delta\hat{\omega}_1(t)$ is itself Gaussian). In general, we can write

$$\langle \delta\hat{\omega}_1(t) \delta\hat{\omega}_1 \rangle = \text{Re} \left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} J(\omega) \right], \quad (3.2.13)$$

where we take the real part since this is the only relevant contribution that will appear in the fidelity. We will consider the case where the spectral density $J(\omega)$ is given by a Lorentzian peaked at a non-zero frequency ν_0 . Consequently, it can be written as

$$J(\omega) = \frac{\sigma^2 \Gamma}{(\omega - \nu_0)^2 + \Gamma^2}. \quad (3.2.14)$$

Here, Γ specifies the width of the Lorentzian and σ specifies its amplitude. Performing the integral in (3.2.13) using (3.2.14), we find that

$$\langle \delta\hat{\omega}_1(t) \delta\hat{\omega}_1 \rangle = \sigma^2 \cos(\nu_0 t) e^{-\Gamma|t|}. \quad (3.2.15)$$

The goal is to compute the fidelity for our state-transfer protocol for the case where the spectral density of the bath degrees-of-freedom is given by the Lorentzian described above. From (3.2.11) and (3.2.12), the relevant quantity to calculate is

$$\langle X^2(t) \rangle = \int_0^t dt_1 \int_0^t dt_2 \cos(2\theta(t_1)) \cos(2\theta(t_2)) \langle \delta\hat{\omega}_1(t_1) \delta\hat{\omega}_1(t_2) \rangle. \quad (3.2.16)$$

Using (3.2.15), this reduces to

$$\langle X^2(t) \rangle = \sigma^2 \int_0^t dt_1 \int_0^t dt_2 \cos(2\theta(t_1)) \cos(2\theta(t_2)) \cos(\nu_0(t_1 - t_2)) e^{-\Gamma|t_1 - t_2|}. \quad (3.2.17)$$

So we see from (3.2.17) that Γ acts as a damping rate. We also need to specify the path for our state transfer protocol in $\{\Omega_1, \Omega_2\}$ space that runs from $t_i = 0$ to t_f . One of the simplest possible paths (and one that will be used throughout this thesis) is a circular path. To achieve this, we choose

$$\Omega_1(t) = -A \sin\left(\frac{n\pi t}{t_f}\right), \quad (3.2.18)$$

$$\Omega_2(t) = A \cos\left(\frac{n\pi t}{t_f}\right), \quad (3.2.19)$$

where n is a parameter that determines how many loops we do during the evolution. Then, from (2.2.19), we get a simple linear relation for the angle $\theta(t)$ given by

$$\theta(t) = \frac{n\pi t}{t_f}. \quad (3.2.20)$$

This choice also keeps the gap $G(t)$ constant. In the ideal case where the damping rate Γ is set to zero, then the integral in (3.2.17) is straightforward to calculate and will lead to some interesting properties for special values of the frequency ν_0 . We remind the reader that ν_0 is the frequency of the peak in the noise spectral density, see equation (3.2.14). Evaluating the integral for a single cycle in parameter space ($n = 1$) and for a total time t_f leads to

$$\langle X^2(t_f) \rangle|_{\Gamma=0, n=1} = \frac{2\sigma^2 t_f^4 \nu_0^2 (1 - \cos(\nu_0 t_f))}{\left(\nu_0^2 t_f^2 - 4\pi^2\right)^2}. \quad (3.2.21)$$

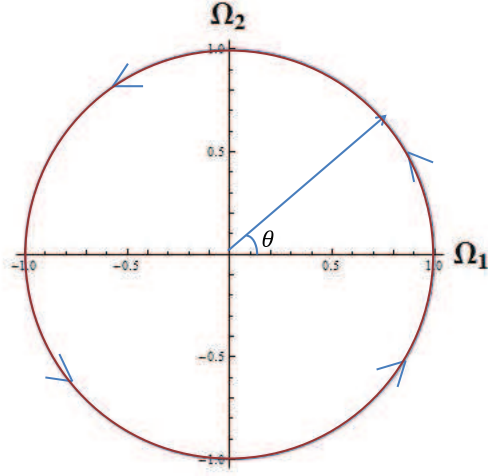


FIGURE 3.2.1. Circular path

Plot of the path for our quantum state transfer protocol in $\{\Omega_1, \Omega_2\}$ space. The functions chosen for our laser amplitudes are given in Eq. (3.2.18) and (3.2.19).

At this stage, one can immediately see that for the values

$$\nu_0 = \frac{2m\pi}{t_f}, \{m \neq 2\} \quad (3.2.22)$$

where m is a positive integer not equal to two, the function $\langle X^2(t_f) \rangle|_{\Gamma=0, n=1}$ is identically zero. Thus, choosing these specific values for the frequency ν_0 (or equivalently, choosing t_f for fixed ν_0) would cancel the effect of having our system coupled to a quantum dissipative bath. It will be instructive to draw plots of $\langle \tilde{\Pi}_{0d}(t) \rangle$, also called coherence, for values of time that start at zero and end at t_f . To obtain these, we integrate equation (3.2.17) from zero to t . From (3.2.1) we would have

$$\langle \tilde{\Pi}_{0d}(t) \rangle = \langle \tilde{\Pi}_{0d}(0) \rangle e^{-i\gamma_d(t)} e^{-\frac{1}{2}\langle X^2(t) \rangle|_{\Gamma=0, n=1}} \quad (3.2.23)$$

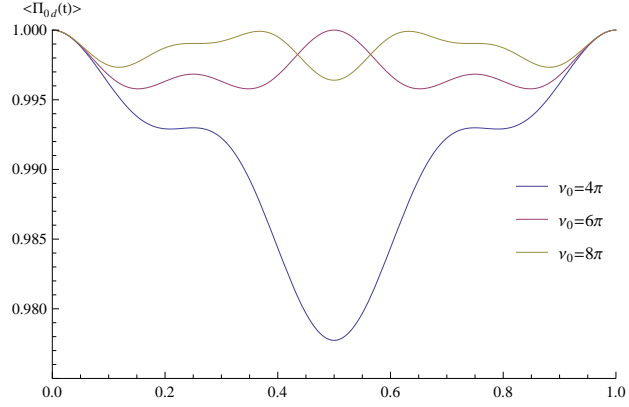


FIGURE 3.2.2. Density matrix revivals

Plot of $\langle \tilde{\Pi}_{0d}(t) \rangle$ as a function of time. We set $t_f = 1$, $\sigma = 1$ and $\Gamma = 0$. Since we chose values of ν_0 which correspond to those of (3.2.22), we observe revivals at the end of the state transfer time. We also set the geometric phase $\gamma_d(t) = 0$.

Since we chose values of ν_0 that correspond to (3.2.22) for the plot of (Fig. 3.2.2), we observe revivals in the relevant component of the density matrix. This means that during the state transfer, our system loses information, and then gains it back at the end of the transfer time. Consequently, if we could engineer a system where the spectral density of the bath degrees of freedom corresponded to a Lorentzian peaked at a non-zero frequency, then we could always choose a frequency given by (3.2.22) that would cancel the effects of a coupling to an environment. Of course if we add the effect of damping ($\Gamma \neq 0$) then this would no longer be the case. In this case we would observe “damped” revivals so instead of having the coherence be unity at the end of the state transfer, it would be smaller by a factor that depends on Γ .

3.3. Environment in thermal equilibrium

Recall that for an environment in thermal equilibrium, the “environment” density matrix takes the form $\hat{\rho}_{env} = \frac{1}{Z_{env}} e^{-\beta \hat{H}_{env}}$. In this section we will consider the case where the

environment mode $\delta\hat{\omega}_1$ can be written as

$$\delta\hat{\omega}_1 = \sum_k \gamma_k \left(\hat{b}_k + \hat{b}_k^\dagger \right), \quad (3.3.1)$$

and that $\hat{H}_{env}(\delta\hat{\omega}_1)$ is quadratic in the bosonic annihilation and creation operators

$$\hat{H}_{env}(\delta\hat{\omega}_1) = \sum_k \epsilon_k \hat{b}_k^\dagger \hat{b}_k. \quad (3.3.2)$$

This system is often referred to as the independent boson model [42]. Given the above constraints it will be possible to apply Wick's theorem to evaluate the average in Eq. (3.2.1).

For simplicity, we define

$$C(t) = \left\langle T_t e^{i\hat{\Theta}(t)} \right\rangle, \quad (3.3.3)$$

where

$$\hat{\Theta}(t) \equiv \int_0^t dt' \cos(2\theta(t')) \delta\hat{\omega}_1(t'). \quad (3.3.4)$$

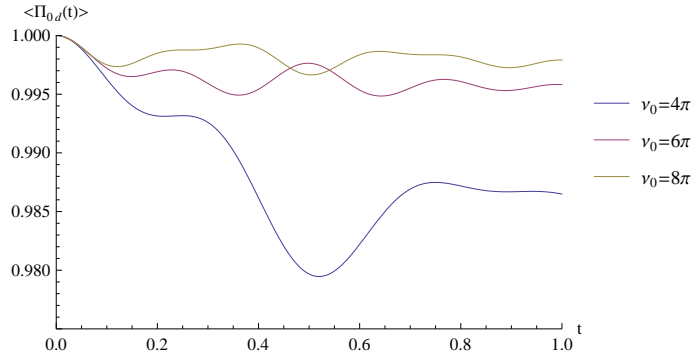


FIGURE 3.2.3. Density matrix revivals with damping

Plot of $\langle \tilde{\Pi}_{0d}(t) \rangle$ as a function of time. We set $t_f = 1$, $\sigma = 1$ and $\Gamma = 1$. Since we chose values of ν_0 which correspond to those of (3.2.22), we observe partial revivals at the end of the state transfer time. We also set the geometric phase $\gamma_d(t) = 0$. In the presence of damping, the coherence does not come back to unity at the end of the state transfer protocol. Some information is inevitably lost.

It will prove to be convenient to expand the exponential in Eq. (3.3.3) in its Taylor series since it will allow us to use Wick's theorem in a convenient way. Doing so, we find

$$C(t) = \sum_{l=0}^{\infty} \frac{(i)^{2l}}{(2l)!} \left\langle T_t \hat{\Theta}^{2l}(t) \right\rangle. \quad (3.3.5)$$

Here we used the fact that only even powers of $\hat{\Theta}(t)$ give non-vanishing terms when performing the time ordered average. Using Wick's theorem, we start with

$$\left\langle T_t \hat{\Theta}^{2k}(t) \right\rangle = [\text{Np}] \left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle^k. \quad (3.3.6)$$

In this case Np represents the number of ways of finding k pairs from $2k$ identical elements. For 4 elements, there would be three of these pairs so that $\text{Np} = 3$. For 6 elements, there would be 5×3 pairs, for 8 there would be $7 \times 5 \times 3$ pairs and so on. In general, for $2k$ pairs we would have

$$\text{Np}_{|2k} = (2k-1)!! = \frac{1}{2^k} \frac{(2k)!}{k!}. \quad (3.3.7)$$

Using this result and (3.3.6), we find

$$C(t) = \sum_k \frac{1}{k!} \left[-\frac{1}{2} \left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle \right]^k, \quad (3.3.8)$$

so that

$$C(t) = e^{-\frac{1}{2} \langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \rangle}. \quad (3.3.9)$$

At this stage we still need to compute $\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle$. Using the definition of $\hat{\Theta}(t)$ we have that

$$\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle = \int_0^t dt_1 \int_0^t dt_2 \cos 2\theta(t_1) \cos 2\theta(t_2) \langle T_t \delta \hat{\omega}_1(t_1) \delta \hat{\omega}_1(t_2) \rangle. \quad (3.3.10)$$

For a circular path in $\{\Omega_1(t), \Omega_2(t)\}$ space, $\theta(t) = -bt$ (where we define $b \equiv \frac{2\pi}{t_f}$) so that (3.3.10) reduces to

$$\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle = \int_0^t dt_1 \int_0^t dt_2 \cos 2bt_1 \cos 2bt_2 \langle T_t \delta \hat{\omega}_1(t_1) \delta \hat{\omega}_1(t_2) \rangle. \quad (3.3.11)$$

We can replace the operator $\delta\hat{\omega}_1$ by its sum over boson creation and annihilation operators (given in Eq. (3.3.1)) into the expression above to find

$$\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle = \sum_k \gamma_k^2 \int_0^t dt_1 \int_0^t dt_2 \cos 2bt_1 \cos 2bt_2 \left[\left\langle T_t \hat{b}_k(t_1) \hat{b}_k^\dagger(t_2) \right\rangle + \left\langle T_t \hat{b}_k^\dagger(t_1) \hat{b}_k(t_2) \right\rangle \right]. \quad (3.3.12)$$

Since the Hamiltonian is quadratic in the bosonic creation and annihilation operators, it is straightforward to obtain their time dependence. From a Heisenberg equation of motion, we can write

$$\dot{\hat{b}}_l(t) = i \left[\hat{H}_{env}, \hat{b}_l(t) \right] = i e^{i\hat{H}_{env}t} \left[\hat{H}_{env}, \hat{b}_l \right] e^{-i\hat{H}_{env}t}. \quad (3.3.13)$$

Using Eq. (3.3.2) the commutator is straightforward to compute and we find

$$\dot{\hat{b}}_l(t) = -i\epsilon_l \hat{b}_l(t), \quad (3.3.14)$$

which has the solution

$$\hat{b}_l(t) = e^{-i\epsilon_l t} \hat{b}_l. \quad (3.3.15)$$

Since we are taking a thermal average of the bosonic operators $\langle \hat{b}_l^\dagger \hat{b}_l \rangle$ for an environment in thermal equilibrium, then we simply have a Bose-Einstein distribution function

$$\langle \hat{b}_l^\dagger \hat{b}_l \rangle = n_B(\epsilon_l) = \frac{1}{e^{\beta\epsilon_l} - 1}. \quad (3.3.16)$$

Using (3.3.15), (3.3.16) and taking explicit consideration of the time ordering operator present in (3.3.9), we find

$$\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle = \sum_k \gamma_k^2 \int_0^t dt_1 \int_0^t dt_2 \cos 2bt_1 \cos 2bt_2 \left[e^{-i\epsilon_k |t_1 - t_2|} + 2n_B(\epsilon_k) \cos(\epsilon_k(t_1 - t_2)) \right]. \quad (3.3.17)$$

We can decompose the contributions into its real and imaginary parts. The real part will be given by

$$\text{Re} \left[\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle \right] = \sum_k \gamma_k^2 (1 + 2n_B(\epsilon_k)) \int_0^t dt_1 \int_0^t dt_2 \cos 2bt_1 \cos 2bt_2 \cos(\epsilon_k(t_1 - t_2)). \quad (3.3.18)$$

Evaluating the integral and using $1 + 2n_B(\epsilon_k) = \cosh\left(\frac{\beta\epsilon_k}{2}\right)$, the real contribution is given by

$$\text{Re} \left[\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle \right] = \sum_k \frac{\gamma_k^2 \cosh\left(\frac{\beta\epsilon_k}{2}\right)}{(4b^2 - \epsilon_k^2)^2} \left[\epsilon_k^2 (1 + \cos(2bt)) \{1 - 2 \cos \epsilon_k t\} - 4\epsilon_k b \sin(2bt) \sin(\epsilon_k t) + 4b^2 \sin^2(4bt) \right]. \quad (3.3.19)$$

Finally, we can also obtain the imaginary contribution

$$\text{Im} \left[\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle \right] = - \sum_k \gamma_k^2 \int_0^t dt_1 \int_0^t dt_2 \cos 2bt_1 \cos 2bt_2 \sin(\epsilon_k |t_1 - t_2|). \quad (3.3.20)$$

Evaluating the integral yields

$$\begin{aligned} \text{Im} \left[\left\langle T_t \hat{\Theta}(t) \hat{\Theta}(t) \right\rangle \right] = & - \sum_k \frac{\gamma_k^2 \epsilon_k}{4b(\epsilon_k^2 - 4b^2)^2} \left[16b^2 \cos(\epsilon_k t) \sin(2bt) \right. \\ & \left. + (\epsilon_k^2 - 4b^2) (4bt + \sin(4bt)) - 8b\epsilon_k \cos(2bt) \sin(\epsilon_k t) \right]. \end{aligned} \quad (3.3.21)$$

We could go on to evaluate these quantities by taking a continuum limit and choosing a specific functional dependence of the coupling strength γ_k . However, the objective of this section was to show how we could apply our methods for computing the coherence of a specific coupling to a bosonic bath. Equation (3.3.19) would give the accumulated phase for our state-transfer protocol which would result in a correction to the closed-system geometric phase in the case where the system would not be coupled to a bosonic bath. In chapter 4 we will apply our methods to the case of an atom coupled to a driven cavity. The noise will arise from fluctuations in the number of photons inside the cavity. The goal will be to perform a state transfer for our qubit and obtain an expression for the fidelity of our state transfer. We will then try to optimize our path in $\{\Omega_1, \Omega_2\}$ space in order to get the best possible fidelity for this particular quantum state transfer.

3.4. Summary

In this section we showed how to obtain the phase of the coherence for the case where our system was coupled to a quantum dissipative bath. We first wrote the Hamiltonian in

a superadiabatic basis by performing a unitary transformation given by Eq. (2.3.3). We proceeded by performing a secular approximation which amounted to throwing away all the off-diagonal terms of the superadiabatic Hamiltonian. This was justified in the limits where $\frac{\dot{\theta}}{G} \ll 1$, $\frac{\dot{\phi}}{G} \ll 1$ and $\frac{|\delta\dot{\omega}_i|}{G} \ll 1$. Since the Hamiltonian was purely diagonal, it was a straightforward matter to find the phase for $\langle \tilde{\Pi}_{0d}(t) \rangle$ by iterating its equation of motion. We then related the fidelity for the quantum state transfer to this phase via Eq. (3.2.9). Using these results, we considered an example where the bath spectral density was given by a Lorentzian peaked at a non-zero frequency. We showed that for the values of the frequency given by Eq. (3.2.22), these gave rise to recurrences in the coherence thus canceling the effects of having our system coupled to a quantum dissipative bath. We concluded this chapter by applying our methods to the independent boson model to show how the state transfer protocol could be used in a physical system.

CHAPTER 4

Atom coupled to a cavity

4.1. Dephasing due to cavity shot noise

In this section we apply the state-transfer methods developed in chapter 3 to a four-level atom coupled to a single cavity mode. To do so, we will drive the cavity with two classical laser fields (also containing a quantum contribution which will add noise to the system) each detuned from the cavity frequency (see figure (4.1.1)). The laser fields will provide the tunable couplings needed for the adiabatic protocol of chapter 3. As was shown in [40], driving the cavity with the two laser fields will induce fluctuations in the number of photons inside the cavity caused by the quantum noise present in each laser field. Consequently, decoherence and dephasing effects will arise when performing our state transfer protocol which will need to be accounted for. Note that this system will be particularly well-suited to performing a phase gate which is another scenario that we will consider at the end of this chapter.

The starting Hamiltonian will have the usual Jaynes-Cummings form with an added contribution arising from the external bath modes and their coupling to the cavity modes. The Hamiltonian thus takes the form

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t), \quad (4.1.1)$$

where we define

$$\hat{H}_0 = \omega_c \hat{a}^\dagger \hat{a} + \omega_{g_1} |g_1\rangle \langle g_1| + \omega_{g_2} |g_2\rangle \langle g_2| + \omega_e |e\rangle \langle e|, \quad (4.1.2)$$

$$\hat{H}_1(t) = g_1 (|e\rangle \langle g_1| \hat{a} + |g_1\rangle \langle e| \hat{a}^\dagger) + g_2 (|e\rangle \langle g_2| \hat{a} + |g_2\rangle \langle e| \hat{a}^\dagger) + \hat{H}'_{env}(t). \quad (4.1.3)$$

Here, ω_c is the frequency associated with the cavity mode and $\hat{H}_{env}(t)$ is given by

$$\begin{aligned} \hat{H}'_{env}(t) = & \sum_q \hbar \omega_q \hat{b}_q^\dagger \hat{b}_q + \omega_c \hat{a}^\dagger \hat{a} + \left(-i\hbar\sqrt{\kappa} \sum_q e^{i\omega_c t} \beta(t) \hat{a} + h.c. \right) \\ & - i\hbar \sqrt{\frac{\kappa}{2\pi\rho}} \sum_q \left[\hat{a}^\dagger \hat{b}_q - \hat{b}_q^\dagger \hat{a} \right] + const. \end{aligned} \quad (4.1.4)$$

which is time-dependent due to the to the classical input field, see (A.0.16). $\hat{H}'_{env}(t)$ must be included since we are considering a driven cavity with one of its ports being partially open (see the discussion in appendix (A)). Ergo, the cavity is being exposed to both the external drive and the vacuum noise so that energy can leak out to the external bath modes. To make a link between the classical drive terms present in the Hamiltonian of (3.1.1), we consider the case where the input field is a coherent drive with a classical and quantum part (see (2.4.28)) which for our state transfer protocol takes the form

$$\hat{b}_{in}(t) = e^{-i\omega_c t} \left(\beta_1(t) + \beta_2(t) + \hat{\xi}(t) \right). \quad (4.1.5)$$

In this case the operator \hat{a} takes the form (in the Heisenberg picture)

$$\hat{a}(t) = \alpha(t) + \hat{d}(t) e^{-i\omega_c t}. \quad (4.1.6)$$

In the above equation, $\alpha(t)$ is the classical cavity amplitude produced by the classical drive tones and \hat{d} is the quantum part. Furthermore, we consider the case where we apply two laser tones on the cavity enabling us to write $\alpha(t)$ as

$$\alpha(t) = \alpha_1(t) e^{-i\omega_1 t} + \alpha_2(t) e^{-i\omega_2 t}, \quad (4.1.7)$$

where $\alpha_1(t)$, $\alpha_2(t)$ are proportional to the complex amplitude of the control lasers and

$$\omega_j = \omega_c + \Delta_j, \quad (4.1.8)$$

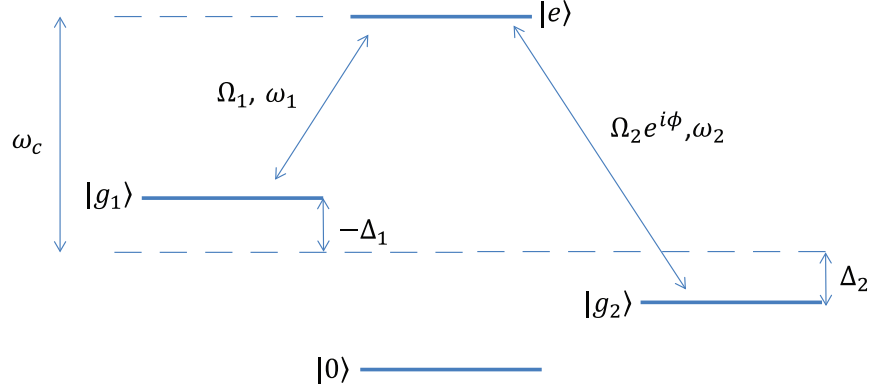


FIGURE 4.1.1. Energy-level diagram in the resonance case
Energy-level diagram showing the structure when the laser tones are resonant with the $(|g_1\rangle, |e\rangle)$ and $(|g_2\rangle, |e\rangle)$ transitions (see (4.1.10)).

with $j \in \{1, 2\}$. Inserting (4.1.6) along with (4.1.7) into (2.4.27), it is straightforward to show that $\alpha_1(t)$ and $\alpha_2(t)$ are related to $\beta_1(t)$ and $\beta_2(t)$ of the input field by

$$\alpha_i(t) = -\sqrt{\kappa} e^{i\Delta_i t} \int_{-\infty}^t e^{\frac{\kappa}{2}(\tau-t)} \beta_i(\tau) d\tau. \quad (4.1.9)$$

The amplitudes $\alpha_1(t)$ and $\alpha_2(t)$ of the classical laser fields will be related to $\Omega_1(t)$ and $\Omega_2(t)$ in equation (4.1.18) and (4.1.19) below. It is important to keep in mind that the displacement transformation performed in (4.1.6) implies that the dynamics of the \hat{d} operator are described in an interaction picture with respect to the $\omega_c \hat{a}^\dagger \hat{a}$ term in the Hamiltonian (4.1.1).

In what follows, we pick the two laser tones to be resonant with the two desired transitions (see figure (4.1.1)) so that

$$\omega_j = \omega_{eg_j} = \omega_c + \Delta_j, \quad (4.1.10)$$

where

$$\omega_{eg_j} = \omega_e - \omega_{g_j}. \quad (4.1.11)$$

When performing the displacement transformation (4.1.6) on the Hamiltonian, there will be terms that give rise to “unwanted” transitions between the ground and excited states.

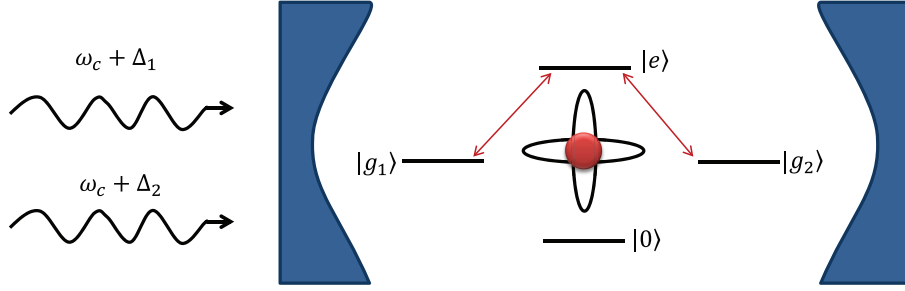


FIGURE 4.1.2. Atom-cavity coupling

Figure representing a four level atom coupled to a single cavity mode. Two laser fields are used to drive the cavity at frequencies $\omega_c + \Delta_1$ and $\omega_c + \Delta_2$. The classical component of the laser fields induce transitions between the levels $(|e\rangle, |g_1\rangle)$ and $(|e\rangle, |g_2\rangle)$. This allows us to transfer the qubit state $\alpha |0\rangle + \beta |g_1\rangle$ to the qubit state $\alpha |0\rangle + \beta e^{i\gamma_d(t_{int})} |g_2\rangle$ and then back to the original qubit state. The fluctuating number of photons inside the cavity will act as the noise terms $\delta\hat{\omega}_1$ and $\delta\hat{\omega}_2$ found in the Hamiltonian (3.1.1).

For example, we can have a term of the form

$$\hat{I}_1 \equiv \alpha_2(t) g_1 e^{-i\omega_2 t} |e\rangle \langle g_1|. \quad (4.1.12)$$

that describes transitions between the levels $|e\rangle, |g_1\rangle$ arising from the second classical laser field driving at the frequency ω_2 . We would like to describe the situation where only the laser field driving at frequency ω_1 creates transitions between the levels $|e\rangle, |g_1\rangle$. Similarly, we only want the field driving at frequency ω_2 to create transitions between the levels $|e\rangle, |g_2\rangle$. If we go into an interaction picture with respect to the Hamiltonian of Eq. (4.1.2), we will have

$$\hat{I}_1 = \alpha_2(t) g_1 e^{-i(\omega_2 - \omega_{eg_1})t} |e\rangle \langle g_1|. \quad (4.1.13)$$

From the resonance condition (4.1.10), we can rewrite (4.1.13) as

$$\hat{I}_1 = \alpha_2(t) g_1 e^{-i(\omega_{g_1} - \omega_{g_2})t} |e\rangle \langle g_1|. \quad (4.1.14)$$

When integrating over time, the terms giving rise to unwanted transitions will scale as

$\frac{\alpha_2(t)g_1}{|\omega_{g_1}-\omega_{g_2}|}$ and $\frac{\alpha_1(t)g_2}{|\omega_{g_1}-\omega_{g_2}|}$. Consequently, if the following conditions are satisfied

$$\frac{\alpha_2(t)g_1}{|\omega_{g_1}-\omega_{g_2}|} \ll 1, \quad (4.1.15)$$

$$\frac{\alpha_1(t)g_2}{|\omega_{g_1}-\omega_{g_2}|} \ll 1, \quad (4.1.16)$$

we could suppress the “unwanted” transitions and so the off-resonant terms in the Hamiltonian could safely be neglected. In what follows, since the magnitude of the detuning frequencies (which later will be taken to be equal magnitude, opposite sign) must be much larger than G , any term of the form

$$\frac{1}{\Delta_i \pm G} \approx \frac{1}{\Delta_i}. \quad (4.1.17)$$

Now, we can write $\alpha_1(t) = i \frac{|\alpha_1(t)|}{2}$, $\alpha_2(t) = i \frac{|\alpha_2(t)|}{2} e^{-i\phi(t)}$ and define

$$\Omega_1(t) \equiv \frac{g_1 |\alpha_1(t)|}{2}, \quad (4.1.18)$$

$$\Omega_2(t) \equiv \frac{g_2 |\alpha_2(t)|}{2}. \quad (4.1.19)$$

With these definitions, the full Hamiltonian of Eq. (4.1.1) in the interaction picture with respect to (4.1.2) along with the resonance condition, (4.1.10), takes the form

$$\begin{aligned} \hat{H}_I(t) = & i \frac{\Omega_1(t)}{2} (|e\rangle \langle g_1| - |g_1\rangle \langle e|) + i \frac{\Omega_2(t)}{2} (e^{-i\phi(t)} |e\rangle \langle g_2| - e^{i\phi(t)} |g_2\rangle \langle e|) \\ & + g_1 (|e\rangle \langle g_1| \hat{d} e^{i\Delta_1 t} + |g_1\rangle \langle e| \hat{d}^\dagger e^{-i\Delta_1 t}) + g_2 (|e\rangle \langle g_2| \hat{d} e^{i\Delta_2 t} + |g_2\rangle \langle e| \hat{d}^\dagger e^{-i\Delta_2 t}) + \hat{H}'_{env}(t). \end{aligned} \quad (4.1.20)$$

Note that due to the resonance condition (4.1.10), only the detuning frequencies are present in the exponent of the interaction picture Hamiltonian. We assume that the experimentalist can control the relative phase $\phi(t)$ between the laser beams. Furthermore, the terms of the form $\alpha^* \hat{d}$ and $\alpha \hat{d}^\dagger$ don't appear if we pick $\alpha(t)$ to solve the classical equation of motion which is obtained from (2.4.22) by keeping only the classical contribution from the displacement transformation. We also ignore the constant $\hbar\omega_c |\alpha|^2$

since it doesn't influence the dynamics of our system and only creates an energy shift.

Without the \hat{d} operator, we have the ideal Hamiltonian studied in chapter 2. The \hat{d} terms include the effects of noise in the cavity, which could generate unwanted transitions between the states $|g_i\rangle$ and $|e\rangle$.

Similar to chapter 3, we are interested in almost-perfect adiabatic evolution where the Ω_j 's are used to tune the wave function of the dark state. We thus want to work in a basis of instantaneous eigenstates of the coherent Hamiltonian of equation (4.1.20). This can be achieved by going into a rotating frame with the unitary operator given by

$$\hat{U} = |0\rangle\langle 0| + |d\rangle\langle d(t)| + |+\rangle\langle +(t)| + |-\rangle\langle -(t)|. \quad (4.1.21)$$

In the first line of (4.1.20), the Ω_j terms are large and \hat{U} will diagonalize this part of the Hamiltonian. The last line in (4.1.20) will describe noise-induced transitions. We remind the reader that following the results of section (3.1), it is crucial to deal with the unwanted effects of cavity noise. This noise can cause virtual transitions from the $|d\rangle$ to $|\pm\rangle$ states and back. Even though these are off resonance, at second order (via virtual transitions) they can cause a dephasing of the dark state which is what we would like to describe. The transitions between the dark and bright states could be due to non-adiabatic corrections ($\dot{\theta}$, $\dot{\phi}$) or due to noise in the cavity (\hat{d} terms). For our particular state-transfer protocol, the dephasing effects result in a reduction of the amplitude of $\langle \tilde{\Pi}_{0d}(t) \rangle$ (see (3.2.9)).

For the discussion that follows, we define

$$\hat{H}_{I,1}(t) \equiv i\frac{\Omega_1(t)}{2}(|e\rangle\langle g_1| - |g_1\rangle\langle e|) + i\frac{\Omega_2(t)}{2}\left(e^{-i\phi(t)}|e\rangle\langle g_2| - e^{i\phi(t)}|g_2\rangle\langle e|\right), \quad (4.1.22)$$

$$\hat{H}_{I,2}(t) \equiv g_1\left(|e\rangle\langle g_1|\hat{d}e^{i\Delta_1 t} + |g_1\rangle\langle e|\hat{d}^\dagger e^{-i\Delta_1 t}\right) + g_2\left(|e\rangle\langle g_2|\hat{d}e^{i\Delta_2 t} + |g_2\rangle\langle e|\hat{d}^\dagger e^{-i\Delta_2 t}\right) + \hat{H}'_{env}(t). \quad (4.1.23)$$

Recall that the instantaneous eigenstates of $\hat{H}_{I,1}(t)$ were given by:

$$|d(t)\rangle = \cos\theta(t)|g_1\rangle + e^{i\phi(t)}\sin\theta(t)|g_2\rangle, \quad (4.1.24)$$

$$|+(t)\rangle = \frac{1}{\sqrt{2}}\left(\sin\theta(t)|g_1\rangle + i|e\rangle - e^{i\phi(t)}\cos\theta(t)|g_2\rangle\right), \quad (4.1.25)$$

$$|- \rangle(t) = \frac{1}{\sqrt{2}} \left(\sin \theta(t) |g_1\rangle - i |e\rangle - e^{i\phi(t)} \cos \theta(t) |g_2\rangle \right). \quad (4.1.26)$$

Where

$$\tan \theta(t) \equiv -\frac{\Omega_1(t)}{\Omega_2(t)}, \quad (4.1.27)$$

$$G(t) \equiv \frac{1}{2} \sqrt{\Omega_1^2(t) + \Omega_2^2(t)}. \quad (4.1.28)$$

We also know that the unitary transformation which diagonalizes $\hat{H}_{I,1}(t)$ in the $\{| \pm \rangle, |d\rangle\}$ basis is given in (4.1.21) with the transformation

$$\hat{H}'(t) = \hat{U} \hat{H}_I(t) \hat{U}^\dagger - i \dot{\hat{U}} \hat{U}^\dagger. \quad (4.1.29)$$

From (4.1.22) and (4.1.21) it can be shown that

$$\hat{U} \hat{H}_{I,1}(t) \hat{U}^\dagger = G(t) \{ |+\rangle \langle +| - |-\rangle \langle -| \}, \quad (4.1.30)$$

and

$$\begin{aligned} -i \hat{U} \dot{\hat{U}}^\dagger &= i \frac{\dot{\theta}}{\sqrt{2}} (|+\rangle \langle d| - |d\rangle \langle +| + |-\rangle \langle d| - |d\rangle \langle -|) - \frac{\dot{\phi}}{\sqrt{2}} (|+\rangle \langle d| + |d\rangle \langle +| + |-\rangle \langle d| + |d\rangle \langle -|) \\ &\quad + \frac{\dot{\phi} \cos^2 \theta(t)}{2} (|+\rangle \langle +| + |-\rangle \langle +| + |+\rangle \langle -| + |-\rangle \langle -|) + \dot{\phi} \sin^2 \theta(t) |d\rangle \langle d|. \end{aligned} \quad (4.1.31)$$

Now we need to calculate $\hat{U} |e\rangle \langle g_1| \hat{U}^\dagger$ and $\hat{U} |e\rangle \langle g_2| \hat{U}^\dagger$ which arise from the transformation of $\hat{H}_{I,2}(t)$. After a bit of math, one can show that

$$\begin{aligned} \hat{U} |e\rangle \langle g_1| \hat{U}^\dagger &= -\frac{i}{\sqrt{2}} \cos \theta(t) (|+\rangle \langle d| - |-\rangle \langle d|) \\ &\quad - \frac{i}{2} \sin \theta(t) (|+\rangle \langle +| + |+\rangle \langle -| - |-\rangle \langle +| - |-\rangle \langle -|), \end{aligned} \quad (4.1.32)$$

and

$$\begin{aligned} \hat{U} |e\rangle \langle g_2| \hat{U}^\dagger &= -\frac{i}{\sqrt{2}} e^{i\phi(t)} \sin \theta(t) (|+\rangle \langle d| - |-\rangle \langle d|) \\ &\quad + \frac{i}{2} e^{i\phi(t)} \cos \theta(t) (|+\rangle \langle +| + |+\rangle \langle -| - |-\rangle \langle +| - |-\rangle \langle -|). \end{aligned} \quad (4.1.33)$$

From this point on we will set both cavity coupling constants to be identical

$$g_1 = g_2 \equiv g. \quad (4.1.34)$$

The condition (4.1.34) is not strictly necessary, but simplifies certain terms and will allow us to obtain much simpler results than if the coupling constants g_1 and g_2 were left completely arbitrary. In order to simplify the notation in what will follow, we define the following operators

$$\hat{F}_1(t) \equiv -\frac{i}{2} \sin \theta(t) g \left(e^{i\Delta_1 t} \hat{d} - e^{-i\Delta_1 t} \hat{d}^\dagger \right), \quad (4.1.35)$$

$$\hat{F}_2(t) \equiv \frac{i}{2} \cos \theta(t) g \left(e^{i\phi(t)} e^{i\Delta_2 t} \hat{d} - e^{-i\phi(t)} e^{-i\Delta_2 t} \hat{d}^\dagger \right), \quad (4.1.36)$$

$$\hat{f}(t) = -\frac{ig}{\sqrt{2}} \left(\cos \theta(t) e^{i\Delta_1 t} + \sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \right) \hat{d}. \quad (4.1.37)$$

The operators in (4.1.35) to (4.1.37) describe various ways noise in \hat{d} affect the system. With these definitions, we can write the Hamiltonian in its instantaneous eigenbasis in the simple form given by

$$\hat{H}'(t) = \hat{H}'_{0,old}(t) + \hat{H}'_{0,new}(t) + \hat{V}(t) + \hat{H}'_{env}. \quad (4.1.38)$$

where the terms in the Hamiltonian are defined as

$$\hat{H}'_{0,old}(t) \equiv G(t) \{ |+\rangle \langle +| - |-\rangle \langle -| \} + \dot{\phi} \sin^2 \theta(t) |d\rangle \langle d| + \frac{1}{2} \dot{\phi} \cos^2 \theta(t) (|+\rangle \langle +| + |-\rangle \langle -|), \quad (4.1.39)$$

$$\hat{H}'_{0,new}(t) \equiv \left(\hat{F}_1(t) + \hat{F}_2(t) \right) (|+\rangle \langle +| - |-\rangle \langle -|), \quad (4.1.40)$$

and

$$\hat{V}(t) \equiv \frac{i}{\sqrt{2}} (\dot{\theta} + i\dot{\phi}) (|+\rangle \langle d| + |-\rangle \langle d|) + \hat{f}(t) (|+\rangle \langle d| - |-\rangle \langle d|) + \hat{V}_{+/-} + h.c., \quad (4.1.41)$$

where $\hat{V}_{+/-}$ gives transitions between the bright states and so will not enter in the dynamics of $\langle \tilde{\Pi}_{0d}(t_f) \rangle$ at leading nontrivial order in a secular approximation. The terms in (4.1.41) that are proportional to $\hat{f}(t)$ and $\hat{f}^\dagger(t)$ will (after performing the Schrieffer-Wolff transformation) give rise to secular noise terms acting on the dark state $|d(t)\rangle$ accounting for virtual transitions $\sim \mathcal{O}(\frac{1}{G})$. This will be understood later on when after performing the

Schrieffer-Wolff transformation, the resulting Hamiltonian will have diagonal components that depend on these noise terms (remember that from (4.1.37), $\hat{f}(t)$ contains operator terms proportional to \hat{d} and \hat{d}^\dagger). Furthermore, it is important to state that we will consider a situation where the cavity noise is narrow-band enough that it cannot cause real (only virtual), energy conserving transitions between the dark and bright states.

At this stage a great simplification can be made. When performing a Schrieffer-Wolff transformation on the above Hamiltonian, it will be necessary to calculate a commutator between an anti-unitary operator $\hat{S}(t)$ with \hat{H}_{env} which would not vanish. In order to avoid having to do this, we will diagonalize \hat{H}_{env} and then go into an interaction picture with respect to the resulting Hamiltonian. To see how this can be achieved, we start by reminding the reader that

$$\hat{H}_{env} = \sum_q \hbar\omega_q \hat{b}_q^\dagger \hat{b}_q - i\hbar\sqrt{\frac{\kappa}{2\pi\rho}} \sum_q \left[\hat{d}^\dagger \hat{b}_q - \hat{b}_q^\dagger \hat{d} \right]. \quad (4.1.42)$$

The next step is to diagonalize both the cavity-bath coupling Hamiltonian and the bath Hamiltonian itself, which is the sum of the two terms in (4.1.42). Doing so, we introduce a new set of normal modes \hat{f}_j (with frequency ω_j) such that

$$\sum_q \hbar\omega_q \hat{b}_q^\dagger \hat{b}_q - i\hbar\sqrt{\frac{\kappa}{2\pi\rho}} \sum_q \left[\hat{d}^\dagger \hat{b}_q - \hat{b}_q^\dagger \hat{d} \right] = \sum_j \hbar\omega_j \hat{f}_j^\dagger \hat{f}_j \quad (4.1.43)$$

so that we have

$$\hat{H}_{env} = \sum_j \hbar\omega_j \hat{f}_j^\dagger \hat{f}_j \quad (4.1.44)$$

We can then write the operator \hat{d} as a linear combination of the new set of normal modes as

$$\hat{d} = \sum_j \lambda_j \hat{f}_j \quad (4.1.45)$$

The λ_j 's are change of basis coefficients, which for our purposes don't need to be explicitly found. Next we go into an interaction picture with respect to the free Hamiltonian of the \hat{f}_j bosons (4.1.44). The \hat{d} operators will now be time-dependent

$$e^{i\hat{H}_{env}t} \hat{d} e^{-i\hat{H}_{env}t} = \hat{d}(t). \quad (4.1.46)$$

The time-dependence we get for $\hat{d}(t)$ is exactly what we would have in the absence of any coupling to the atom and any classical drives. We can thus get the properties of $\hat{d}(t)$ by solving the Heisenberg Langevin equations for the uncoupled cavity, as was done in equation (2.4.22) in section (2.4). Therefore, $\hat{d}(t)$ will have the standard noise properties we expect given by (2.4.36).

Now we can perform a Schrieffer-Wolff transformation on the Hamiltonian in (4.1.38) to find a simplified effective Hamiltonian accounting for virtual transitions due to leading-order non-secular terms. The terms $\hat{f}(t)(|+\rangle\langle d| - |-\rangle\langle d|) + \hat{f}^\dagger(t)(|d\rangle\langle +| - |d\rangle\langle -|)$ in (4.1.41) are the ones which will play an important role since after performing the Schrieffer-Wolff transformation, they will give rise to noise terms proportional to $|d\rangle\langle d|$ and as we saw in (3.1.18) and (3.2.1) these terms enter directly into the exponent of $\langle \tilde{\Pi}_{0d}(t) \rangle$. We start by performing the time-dependent canonical transformation given by

$$\begin{aligned} \tilde{H}'(t) &= e^{\hat{S}(t)} \hat{H}'(t) e^{-\hat{S}(t)} - ie^{\hat{S}(t)} \frac{\partial}{\partial t} e^{-\hat{S}(t)} \\ &= \hat{H}'(t) + [\hat{S}(t), \hat{H}'(t)] + \frac{1}{2} [\hat{S}(t), [\hat{S}(t), \hat{H}'(t)]] - ie^{\hat{S}(t)} \frac{\partial}{\partial t} e^{-\hat{S}(t)} + \mathcal{O}(\hat{S}(t)^3) \end{aligned} \quad (4.1.47)$$

Note that in writing (4.1.47), $[\hat{S}(t), \dot{\hat{S}}(t)] \neq 0$. Consequently, one should be careful when expanding the right hand side of (4.1.47) in powers of $\hat{S}(t)$. We only keep leading order terms in $\hat{S}(t)$ because as we show below in (4.1.82) and the definition of the coefficients, $\hat{S}(t) \sim \mathcal{O}\left(\frac{\|\hat{V}\|}{G}, \frac{\|\hat{V}\|}{\Delta}\right)$ and higher order terms in $\left(\frac{\|\hat{V}\|}{G}, \frac{\|\hat{V}\|}{\Delta}\right)$ are neglected. Then we can approximate

$$-ie^{\hat{S}(t)} \frac{\partial}{\partial t} e^{-\hat{S}(t)} = i \frac{d\hat{S}(t)}{dt} + \frac{i}{2} \left[\hat{S}(t), \frac{d\hat{S}(t)}{dt} \right] + \mathcal{O}\left(\frac{\|\dot{\hat{V}}\| \|\hat{V}\|^2}{G^3}\right) \quad (4.1.48)$$

In what follows, we will approximate the last term in (4.1.47) by the first two terms in (4.1.48). To eliminate the non-secular terms of order $\dot{\theta}$, $\dot{\phi}$ and g , $\hat{S}(t)$ is chosen to satisfy

$$[\hat{S}(t), \hat{H}'_{0,old}(t)] + \hat{V}(t) + i \frac{d\hat{S}(t)}{dt} = 0. \quad (4.1.49)$$

We don't include $\hat{H}'_{0,new}(t)$ into the commutator of (4.1.49) since $\hat{H}'_{0,old}(t) \sim G(t)$ will scale much larger than $\hat{F}_1(t) \sim g\sqrt{N_{th}}$ and $\hat{F}_2(t)$. Consequently, its inclusion would not affect the solution to the differential equation in (4.1.49). The only role of $\hat{H}'_{0,new}(t)$ is that it can cause the energy of the $|\pm\rangle$ states to wiggle. Expanding $-ie^{\hat{S}(t)}\frac{\partial}{\partial t}e^{-\hat{S}(t)}$ as in (4.1.48) and inserting the result into (4.1.47), we find that

$$\begin{aligned}\tilde{\hat{H}}'(t) &= \hat{H}'_{0,old}(t) + \hat{H}'_{0,new}(t) + \hat{V}(t) + i\frac{d\hat{S}(t)}{dt} + \frac{i}{2}\left[\hat{S}(t), \frac{d\hat{S}(t)}{dt}\right] \\ &+ \left[\hat{S}(t), \hat{H}'_{0,old}(t) + \hat{H}'_{0,new}(t) + \hat{V}(t)\right] \\ &+ \frac{1}{2}\left[\hat{S}(t), \left[\hat{S}(t), \hat{H}'_{0,old}(t) + \hat{H}'_{0,new}(t) + \hat{V}(t)\right]\right] + \mathcal{O}\left(G\hat{S}(t)^3, g\sqrt{N_{th}}\hat{S}(t)^3, \hat{S}^2(t)\frac{d\hat{S}(t)}{dt}\right)\end{aligned}\quad (4.1.50)$$

The factor of $g\sqrt{N_{th}}$ in the last term of (4.1.50) comes from the size of $\hat{H}'_{0,new}(t)$. Using (4.1.49), and neglecting terms of $\mathcal{O}\left(G\hat{S}(t)^3, g\sqrt{N_{th}}\hat{S}(t)^3, \hat{S}^2(t)\frac{d\hat{S}(t)}{dt}\right)$, (4.1.50) becomes

$$\begin{aligned}\tilde{\hat{H}}'(t) &\cong \hat{H}'_{0,old}(t) + \hat{H}'_{0,new}(t) + \left[\hat{S}(t), \hat{H}'_{0,new}(t)\right] \\ &+ \frac{1}{2}\left[\hat{S}(t), \hat{V}(t)\right] + \mathcal{O}\left(g\sqrt{N_{th}}\hat{S}(t)^2, \hat{S}^2(t)\frac{d\hat{S}(t)}{dt}\right),\end{aligned}\quad (4.1.51)$$

where by choosing $\hat{S}(t)$ as per equation (4.1.49) (which depends explicitly on $\frac{d\hat{S}(t)}{dt}$), the $\left[\hat{S}(t), \frac{d\hat{S}(t)}{dt}\right]$ commutator term arising from the transformation gets exactly canceled. After finding the explicit form of $\hat{S}(t)$, we will be able to determine the typical size of the sub-leading terms that were dropped when going from (4.1.50) to (4.1.51) (i.e. terms of order $\mathcal{O}\left(g\sqrt{N_{th}}\hat{S}(t)^2, \hat{S}^2(t)\frac{d\hat{S}(t)}{dt}\right)$) self-consistently. This will be done in detail in the next subsection.

In order to find the anti-unitary operator $\hat{S}(t)$ that satisfies (4.1.49), we make the following ansatz:

$$\hat{S}(t) = \hat{\chi}_1(t)|+\rangle\langle d| + \hat{\chi}_2(t)|-\rangle\langle d| + \hat{\chi}_3(t)|+\rangle\langle -| - h.c. \quad (4.1.52)$$

However, in what follows, we will drop the $\hat{\chi}_3(t)$ term since it will not contribute to $\langle \tilde{\Pi}_{0d}(t_f) \rangle$ at leading nontrivial order in the secular approximation. We include a “hat” on the coefficients of $\hat{S}(t)$ since these will be operators. Enforcing that $\hat{S}(t)$ satisfies (4.1.49), the coefficients $\hat{\chi}_i(t)$ ’s are found to be given by the differential equations

$$\hat{\chi}_1(t) \left(\dot{\phi} \sin^2 \theta(t) - G(t) - \frac{1}{2} \dot{\phi} \cos^2 \theta(t) \right) = -\frac{1}{\sqrt{2}} (i\dot{\theta} - \dot{\phi}) - \hat{f}(t) - i \frac{d}{dt} \hat{\chi}_1(t), \quad (4.1.53)$$

$$\hat{\chi}_2(t) \left(\dot{\phi} \sin^2 \theta(t) + G(t) - \frac{1}{2} \dot{\phi} \cos^2 \theta(t) \right) = -\frac{1}{\sqrt{2}} (i\dot{\theta} - \dot{\phi}) + \hat{f}(t) - i \frac{d}{dt} \hat{\chi}_2(t), \quad (4.1.54)$$

We are interested in the adiabatic regime hence the rate at which $\theta(t)$ and $\phi(t)$ change is much, much smaller than $G(t)$. This allows us to make the following approximation

$$\dot{\phi} \sin^2 \theta(t) \pm G(t) - \frac{1}{2} \dot{\phi} \cos^2 \theta(t) \approx \pm G(t). \quad (4.1.55)$$

The above approximation allows us to rewrite the differential equation in (4.1.53) as

$$\frac{d}{dt} \hat{\chi}_1(t) + iG(t) \hat{\chi}_1(t) = i \left[\frac{1}{\sqrt{2}} (i\dot{\theta} - \dot{\phi}) + \hat{f}(t) \right]. \quad (4.1.56)$$

Note that (4.1.56) is just the equation of motion of an undamped, driven simple harmonic oscillator with a time-dependent frequency $G(t)$. For simplification, we define the operator

$$\hat{P}(t) \equiv \frac{1}{\sqrt{2}} (i\dot{\theta} - \dot{\phi}) + \hat{f}(t). \quad (4.1.57)$$

which is just like the driving force on the oscillator. Then, the general solution to the differential equation in (4.1.56) is given by

$$\hat{\chi}_1(t) = i e^{-i \int_0^t G(t') dt'} \int_0^t e^{i \int_0^{t'} G(t'') dt''} \hat{P}(t') dt'. \quad (4.1.58)$$

For simplicity, we assume that $G(t)$ is time independent. This can be achieved by using equations (3.2.18) and (3.2.19) of chapter three

$$\Omega_1 = -A \sin \theta(t), \quad (4.1.59)$$

and

$$\Omega_2 = A \cos \theta(t). \quad (4.1.60)$$

where A is a constant amplitude. Consequently, (4.1.58) can be written as

$$\hat{\chi}_1(t) = ie^{-iGt} \int_0^t e^{iGt'} \hat{P}(t') dt'. \quad (4.1.61)$$

Inserting the definition of $\hat{P}(t)$ into (4.1.61), we can decompose it into two parts,

$$\begin{aligned} \hat{\chi}_1(t) &= \frac{i}{\sqrt{2}} e^{-iGt} \int_0^t e^{iGt'} (i\dot{\theta}(t') - \dot{\phi}(t')) dt' \\ &\quad + ie^{-iGt} \int_0^t e^{iGt'} \hat{f}(t') dt'. \end{aligned} \quad (4.1.62)$$

We begin by focusing on the first line in (4.1.62) and define the function

$$h(t) \equiv i\dot{\theta}(t) - \dot{\phi}(t). \quad (4.1.63)$$

The natural frequency of our effective simple harmonic oscillator is G . The functions $\dot{\theta}(t)$ and $\dot{\phi}(t)$ are very slow. Thus, the $h(t)$ term in (4.1.63) is like driving our simple harmonic oscillator with a frequency that is very far from resonance so that we expect a very weak response. We can use this fact to greatly simplify the first term in (4.1.62). We start by writing

$$\hat{\chi}_1 = \hat{\chi}_{1,1} + \hat{\chi}_{1,2}, \quad (4.1.64)$$

with

$$\hat{\chi}_{1,1}(t) \equiv \frac{i}{\sqrt{2}} e^{-iGt} \int_0^t e^{iGt'} h(t') dt' \quad (4.1.65)$$

and $\hat{\chi}_{1,2}(t)$ defined in (4.1.71) below. We can integrate (4.1.65) by parts to find that

$$\hat{\chi}_{1,1}(t) = \frac{1}{\sqrt{2}G} [h(t) - e^{-iGt} h(0)] + \mathcal{O}\left(\frac{\ddot{\theta}}{G^2}, \frac{\ddot{\phi}}{G^2}\right) \quad (4.1.66)$$

Next we also need to consider the second term in (4.1.62). Recall that the function $\hat{f}(t)$ was defined as

$$\hat{f}(t) = -\frac{ig}{\sqrt{2}} \left(\cos \theta(t) e^{i\Delta_1 t} + \sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \right) \hat{d}(t). \quad (4.1.67)$$

With this in mind we define

$$\hat{\chi}_{1,2}(t) \equiv -\frac{ig}{\sqrt{2}} e^{-iGt} \int_0^t e^{iGt'} \left(\cos \theta(t) e^{i\Delta_1 t} + \sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \right) \hat{d}(t') dt'. \quad (4.1.68)$$

We can use the same arguments that lead from (4.1.65) to (4.1.66) in order to calculate (4.1.68). Considering the case where the noise stemming from $\hat{d}(t)$ is slow, meaning that the cavity-damping rate is much smaller than the detuning frequency, we define the operators

$$\hat{v}_1(t) = \cos \theta(t) \hat{d}(t), \quad (4.1.69)$$

$$\hat{v}_2(t) = \sin \theta(t) e^{i\phi(t)} \hat{d}(t), \quad (4.1.70)$$

and integrate by parts to find that

$$\hat{\chi}_{1,2}(t) = -\frac{g}{\sqrt{2}} \left[\frac{e^{i\Delta_1 t} \hat{v}_1(t) - e^{-iGt} \hat{d}}{\Delta_1} + \frac{e^{i\Delta_2 t} \hat{v}_2(t)}{\Delta_2} \right] + \mathcal{O}\left(\frac{g\dot{v}_1}{\Delta_1^2}, \frac{g\dot{v}_2}{\Delta_2^2}\right), \quad (4.1.71)$$

where we used the fact that $|\Delta_i| \gg G$ to simplify the denominator of the above term. To neglect the higher order terms in both (4.1.66) and (4.1.71), we require that

$$\frac{\text{Max}(\ddot{\theta}, \ddot{\phi})}{G^2} \ll 1, \quad (4.1.72)$$

$$\frac{\text{Max}(\dot{\theta}, \dot{\phi})}{\Delta_i} \ll 1, \quad (4.1.73)$$

which are consistent with the adiabatic criteria. To obtain an estimate for the size of the $\dot{\hat{d}}(t)$ term, we remind the reader that from section (2.4), we have

$$\hat{d}(t) = -\sqrt{\kappa} \int_{-\infty}^t d\tau e^{-\kappa/2(t-\tau)} \hat{\xi}(\tau). \quad (4.1.74)$$

Notice that the detuning frequency does not appear in our chosen frame (see (4.1.6)). We also remind the reader that the field $\hat{\xi}(t)$ satisfies

$$\langle \hat{\xi}^\dagger(t) \hat{\xi}(t') \rangle = N_{th} \delta(t - t'). \quad (4.1.75)$$

Using (4.1.74) and (4.1.75), it is straightforward to show that

$$\langle \hat{d}^\dagger(t) \hat{d}(t) \rangle = N_{th}, \quad (4.1.76)$$

and

$$\langle \dot{\hat{d}}^\dagger(t) \dot{\hat{d}}(t) \rangle = -\frac{3}{4} N_{th} \kappa^2. \quad (4.1.77)$$

Thus we conclude that

$$\sqrt{\left| \frac{\langle \dot{\hat{d}}^\dagger(t) \dot{\hat{d}}(t) \rangle}{\langle \hat{d}^\dagger(t) \hat{d}(t) \rangle} \right|} \sim \kappa. \quad (4.1.78)$$

Going back to (4.1.71) and using (4.1.78), we also require the condition

$$\frac{\kappa}{\Delta_i} \ll 1, \quad (4.1.79)$$

to be satisfied in order to neglect the higher order terms in (4.1.71). Using (4.1.66), (4.1.71)

and the initial condition $\theta(0) = 0$, we find that

$$\hat{\chi}_1(t) = \frac{1}{\sqrt{2}G} [h(t) - e^{-iGt}h(0)] - i\frac{g}{\sqrt{2}} \left[\frac{e^{i\Delta_1 t} \cos \theta(t) \hat{d}(t) - e^{-iGt} \hat{d}}{\Delta_1} + \frac{\sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \hat{d}(t)}{\Delta_2} \right]. \quad (4.1.80)$$

From (4.1.54), we can perform the same set of approximations that we used to obtain $\hat{\chi}_1(t)$, and we find that $\hat{\chi}_2(t)$ is obtained from $\hat{\chi}_1(t)$ by replacing $G \rightarrow -G$ and $\hat{f}(t) \rightarrow -\hat{f}(t)$. The physical origin of these differences is as follows: $\hat{\chi}_{1/2}(t)$ correspond to virtual transitions (respectively) to the $|\pm\rangle$ state. The energy of the virtual state is different, as is the relevant matrix element. The origin of the sign difference can be traced back in the matrix element of the form of $\hat{V}(t)$ in the instantaneous eigenstate basis (see (4.1.41)). Thus we have

$$\hat{\chi}_2(t) = -\frac{1}{\sqrt{2}G} [h(t) - e^{iGt}h(0)] + i\frac{g}{\sqrt{2}} \left[\frac{e^{i\Delta_1 t} \cos \theta(t) \hat{d}(t) - e^{iGt} \hat{d}}{\Delta_1} + \frac{\sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \hat{d}(t)}{\Delta_2} \right]. \quad (4.1.81)$$

The anti-unitary operator $\hat{S}(t)$ (omitting the $\hat{\chi}_3$ term) can then be written as

$$\hat{S}(t) = \hat{\chi}_1(t) |+\rangle \langle d| + \hat{\chi}_2(t) |-\rangle \langle d| - h.c. \quad (4.1.82)$$

with $\hat{\chi}_1(t)$ given by (4.1.80) and $\hat{\chi}_2(t)$ by (4.1.81).

Next, we need to calculate the commutator that will give rise to the term proportional to $|d\rangle\langle d|$ in (4.1.51) $(\frac{1}{2}[\hat{S}(t), \hat{V}(t)])$. Writing only the $|d\rangle\langle d|$ term, it is found to be

$$\begin{aligned} \frac{1}{2}[\hat{S}(t), \hat{V}(t)] &= \frac{1}{2} \left[\frac{(\dot{\phi} + i\dot{\theta})}{\sqrt{2}} (\hat{\chi}_1(t) + \hat{\chi}_2(t)) + \frac{(\dot{\phi} - i\dot{\theta})}{\sqrt{2}} (\hat{\chi}_1^\dagger(t) + \hat{\chi}_2^\dagger(t)) \right. \\ &\quad \left. + \hat{f}^\dagger(t) (\hat{\chi}_2(t) - \hat{\chi}_1(t)) + (\hat{\chi}_2^\dagger(t) - \hat{\chi}_1^\dagger(t)) \hat{f}(t) \right] |d\rangle\langle d| \end{aligned} \quad (4.1.83)$$

Note that since the function $h(t)$ has the opposite sign in $\hat{\chi}_1(t)$ than in $\hat{\chi}_2(t)$ it disappears from the first line of (4.1.83). This means that there will be no new terms of order $(\dot{\theta} + \dot{\phi})^2$ which modify the energy of the $|d\rangle$ state.

Before proceeding further, we can perform a simplification that will greatly reduce the complexity of the resulting Hamiltonian. When calculating $\langle \tilde{\Pi}_{0d}(t_f) \rangle$, we will have to integrate the term proportional to $|d\rangle\langle d|$ over time. Terms that oscillate with the phase $e^{i(G \pm \Delta_i)t}$ or e^{iGt} will scale as (for a sufficiently slow-varying function $f(t)$)

$$\int_0^{t_f} f(t) e^{iGt} dt = \frac{1}{iG} \left[f(t_f) e^{iGt_f} - f(0) - \mathcal{O}\left(\frac{\dot{f}}{G}\right) \right], \quad (4.1.84)$$

whereas

$$\int_0^{t_f} f(t) dt = f(t_f) t_f - \int_0^{t_f} \dot{f}(t) t dt. \quad (4.1.85)$$

Given that the adiabatic criteria require that $Gt_f \gg 1$ and $|G \pm \Delta_i| t_f \gg 1$, this guarantees that we can safely drop terms with oscillating phases $e^{i(G \pm \Delta_i)t}$ and e^{iGt} in (4.1.83). An immediate consequence is that the term in the first line of (4.1.83) can be dropped. The reason is that the sum of $\hat{\chi}_1(t) + \hat{\chi}_2(t)$ has a resulting contribution proportional to the phases $e^{\pm i\Delta_j t}$. Physically, this means that we can just add the effective energy shifts associated with each perturbation that gives us transitions to the $|\pm\rangle$ states. For the second line in (4.1.83), we can use (4.1.80) and (4.1.81) for the operators $\hat{\chi}_1(t)$, $\hat{\chi}_2(t)$ as well as (4.1.67) for the function $\hat{f}(t)$ to show that (keeping terms with no oscillating phase)

$$\int_0^t [\hat{f}^\dagger(t') \hat{\chi}_1(t') + \hat{\chi}_1^\dagger(t') \hat{f}(t')] dt' \cong g^2 \int_0^t \left[\frac{\cos^2 \theta(t')}{\Delta_1} + \frac{\sin^2 \theta(t')}{\Delta_2} \right] \hat{d}^\dagger(t') \hat{d}(t') dt' \quad (4.1.86)$$

and

$$\int_0^t \left[\hat{f}^\dagger(t') \hat{\chi}_2(t') + \hat{\chi}_2^\dagger(t') \hat{f}(t') \right] dt' \cong -g^2 \int_0^t \left[\frac{\cos^2 \theta(t')}{\Delta_1} + \frac{\sin^2 \theta(t')}{\Delta_2} \right] \hat{d}^\dagger(t') \hat{d}(t') dt' \quad (4.1.87)$$

We thus conclude that

$$\frac{1}{2} [\hat{S}(t), \hat{V}(t)] \cong -g^2 \left[\frac{\cos^2 \theta(t)}{\Delta_1} + \frac{\sin^2 \theta(t)}{\Delta_2} \right] \hat{d}^\dagger(t) \hat{d}(t) \quad (4.1.88)$$

Performing a secular approximation so that we keep only the diagonal terms in the transformed Hamiltonian and writing only the dephasing term proportional to $|d\rangle\langle d|$, we find that

$$\begin{aligned} \tilde{H}'(t) \approx & G(t) \{ |+\rangle\langle +| - |-\rangle\langle -| \} + \dot{\phi} \sin^2 \theta(t) |d\rangle\langle d| + \\ & + \frac{1}{2} \dot{\phi} \cos^2 \theta(t) (|+\rangle\langle +| + |-\rangle\langle -|) - g^2 \left\{ \frac{\cos^2 \theta(t)}{\Delta_1} + \frac{\sin^2 \theta(t)}{\Delta_2} \right\} \hat{d}^\dagger(t) \hat{d}(t) |d\rangle\langle d| \\ & + \text{other} \end{aligned} \quad (4.1.89)$$

To make further progress, we consider the case where the two detuning frequencies are equal and opposite in magnitude

$$\Delta = \Delta_1 = -\Delta_2. \quad (4.1.90)$$

In this case the Hamiltonian of (4.1.89) then becomes

$$\begin{aligned} \tilde{H}'(t) \approx & G(t) \{ |+\rangle\langle +| - |-\rangle\langle -| \} + \dot{\phi} \sin^2 \theta(t) |d\rangle\langle d| + \\ & + \frac{1}{2} \dot{\phi} \cos^2 \theta(t) (|+\rangle\langle +| + |-\rangle\langle -|) - \frac{g^2}{\Delta} \cos 2\theta(t) \hat{d}^\dagger(t) \hat{d}(t) |d\rangle\langle d| \\ & + \text{other}, \end{aligned} \quad (4.1.91)$$

where the other terms in (4.1.91) are proportional to $|+\rangle\langle +|$ and $|-\rangle\langle -|$ which don't influence the dynamics of the coherence. Using (4.1.91), we finally have

$$\left\langle \tilde{\Pi}_{0d}(t) \right\rangle = \alpha \beta^* e^{-i\gamma_d(t)} \left\langle T_t e^{i\frac{g^2}{\Delta} \int_0^t \cos 2\theta(t') \hat{d}^\dagger(t') \hat{d}(t') dt'} \right\rangle. \quad (4.1.92)$$

It is important to keep in mind that in order for the above expression to hold, we need to assume that the laser detunings are much larger than the amplitude G and that the changes to the initial state $\hat{\rho}(0)$ are negligible (so that only the $|d\rangle\langle d|$ in the Hamiltonian of (4.1.91) is relevant for the phase of $\langle \tilde{\Pi}_{0d}(t) \rangle$). In the next section we will show that indeed the changes to the initial state under the limits that we considered will not affect the dynamics of our system so that we can safely use (4.1.92) to describe the fidelity of our state transfer protocol.

4.1.1. A note on $g_1 \neq g_2$. Recall that the Hamiltonian of equation (4.1.91) was derived in the regime where $g_1 = g_2 = g$ and $\Delta_1 = -\Delta_2$. As it turns out, it is also possible to obtain an expression analogous to (4.1.91) without requiring that both cavity-coupling constants to be identical. To see this, we go back to the definition of the operator $\hat{f}(t)$ first written in (4.1.37). If we let both cavity-coupling constants to be independent of each other, it is easy to verify that in this case $\hat{f}(t)$ becomes

$$\hat{f}(t) = -\frac{i}{\sqrt{2}} \left(g_1 \cos \theta(t) e^{i\Delta_1 t} + g_2 \sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \right) \hat{d}. \quad (4.1.93)$$

Following the same steps that led to equations (4.1.68) and (4.1.71), the operators $\hat{\chi}_1(t)$ and $\hat{\chi}_2(t)$ now become

$$\hat{\chi}_1(t) = \frac{1}{\sqrt{2}G} [h(t) - e^{-iGt}h(0)] - i\frac{1}{\sqrt{2}} \left[\frac{g_1 \left(e^{i\Delta_1 t} \cos \theta(t) \hat{d}(t) - e^{-iGt} \hat{d} \right)}{\Delta_1} + \frac{g_2 \sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \hat{d}(t)}{\Delta_2} \right], \quad (4.1.94)$$

$$\hat{\chi}_2(t) = -\frac{1}{\sqrt{2}G} [h(t) - e^{iGt}h(0)] + i\frac{g}{\sqrt{2}} \left[\frac{g_1 \left(e^{i\Delta_1 t} \cos \theta(t) \hat{d}(t) - e^{iGt} \hat{d} \right)}{\Delta_1} + \frac{g_2 \sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \hat{d}(t)}{\Delta_2} \right]. \quad (4.1.95)$$

Then following the same steps of (4.1.86), (4.1.87) and (4.1.88), the Hamiltonian becomes

$$\begin{aligned}\tilde{H}'(t) \approx & G(t) \{ |+\rangle \langle +| - |-\rangle \langle -| \} + \dot{\phi} \sin^2 \theta(t) |d\rangle \langle d| + \\ & + \frac{1}{2} \dot{\phi} \cos^2 \theta(t) (|+\rangle \langle +| + |-\rangle \langle -|) - \left\{ \frac{g_1^2 \cos^2 \theta(t)}{\Delta_1} + \frac{g_2^2 \sin^2 \theta(t)}{\Delta_2} \right\} \hat{d}^\dagger(t) \hat{d}(t) |d\rangle \langle d| \\ & + \text{other}\end{aligned}\tag{4.1.96}$$

Consequently, we can obtain the same $\cos 2\theta(t)$ dependence as in (4.1.91) by setting

$$\frac{g_1^2}{\Delta_1} = -\frac{g_2^2}{\Delta_2}\tag{4.1.97}$$

The condition (4.1.97) is much less restrictive than requiring that the cavity-coupling constants be identical and the detuning frequencies be equal in magnitude with opposite sign.

4.1.2. Estimates for the size of $\hat{S}(t)$. As was mentioned in the paragraph below (4.1.51), we now determine the size for the terms that were neglected by going from (4.1.50) to (4.1.51). Recall that we found (omitting the $\hat{\chi}_3$ term)

$$\hat{S}(t) = \hat{\chi}_1(t) |+\rangle \langle d| + \hat{\chi}_2(t) |-\rangle \langle d| - h.c.\tag{4.1.98}$$

with

$$\hat{\chi}_1(t) = \frac{1}{\sqrt{2}G} (i\dot{\theta}(t) - \dot{\phi}(t)) - i\frac{g}{\sqrt{2}} \left[\frac{e^{i\Delta_1 t} \cos \theta(t) \hat{d}(t)}{\Delta_1} + \frac{\sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \hat{d}(t)}{\Delta_2} \right],\tag{4.1.99}$$

We don't bother writing down $\hat{\chi}_2(t)$ since it scales the same way as $\hat{\chi}_1(t)$. When we performed the Schrieffer-Wolff transformation, the terms we threw away in (4.1.51) scaled as

$$\mathcal{O} \left(G \hat{S}^3(t), g \sqrt{N_{th}} \hat{S}(t)^2, \hat{S}^2(t) \frac{d\hat{S}(t)}{dt} \right),\tag{4.1.100}$$

where the factor of $g\sqrt{N_{th}}$, being the size of $\hat{H}'_{0,new}(t)$ (see (4.1.40) along with (4.1.35) and (4.1.36)) arises from the term $[\hat{S}(t), [\hat{S}(t), \hat{H}'_{0,new}(t)]]$. The terms we keep in (4.1.51) are

of the order $\mathcal{O}\left(g\sqrt{N_{th}}\hat{S}(t), \frac{d\hat{S}(t)}{dt}\right)$. Now, the operator $\frac{d\hat{S}(t)}{dt}$ will scale as

$$\frac{d\hat{S}(t)}{dt} \sim \text{Max} \left\{ \frac{\ddot{\theta}}{G}, \frac{\ddot{\phi}}{G}, g, \frac{g\dot{d}}{\Delta_{1,2}}, \frac{g\dot{\theta}\sqrt{N_{th}}}{\Delta_{1,2}}, \frac{g\dot{\phi}\sqrt{N_{th}}}{\Delta_2}, \right\} \quad (4.1.101)$$

Using the same reasoning that led to (4.1.78), we summarize in a table the conditions that have to be satisfied so that the terms in (4.1.100) can be safely neglected and that give rise to (4.1.92):

Adiabatic criteria	$\frac{\dot{\phi}}{G} \ll 1; \frac{\dot{\theta}}{G} \ll 1; \frac{1}{Gt_f} \ll 1$
Adiabatic criteria (2)	$\frac{\text{Max}(\ddot{\theta}, \ddot{\phi})}{G^2} \ll 1$
Small coupling	$\frac{g\sqrt{N_{th}}}{G} \ll 1$
Large detuning	$\frac{G}{\Delta} \ll 1$
Small cavity damping rate	$\frac{\kappa}{\Delta} \ll 1$
Adiabatic/large detuning criteria	$\frac{g\text{Max}(\dot{\theta}, \dot{\phi})}{\Delta^2} \ll 1$
Detuning condition	$\Delta = \Delta_1 = -\Delta_2$

Table 1: List of conditions for the range of validity of our state transfer protocol.

We add a last note on the upper bound of t_f . Recall that in section (2.1), we showed that the adiabatic approximation was only valid for time scales such that

$$\int_0^t \sum_n c_n(t') e^{i[\theta_n(t') - \theta_m(t')]} \frac{\langle \phi_m(t') | \dot{\hat{H}}(t') | \phi_n(t') \rangle}{[E_n(t') - E_m(t')]} dt' \lesssim 1, \quad (4.1.102)$$

where $\theta_n(t)$ is the dynamical phase of the instantaneous eigenstate $|\phi_n(t)\rangle$ of the Hamiltonian. If we use only the first line in (4.1.20), then we can show that

$$\langle -(t) | \dot{\hat{H}}(t) | +(t) \rangle \sim G \text{Max} \{ \dot{\theta}, \dot{\phi} \} \quad (4.1.103)$$

Using the fact that the eigenvalues of the bright states are $E_{\pm} = \pm G$ and that $\{\dot{\theta}, \dot{\phi}\} \sim \frac{1}{t_f}$, then it is straightforward to show that

$$\int_0^t \sum_n c_n(t') e^{i[\theta_n(t') - \theta_m(t')]} \frac{\langle \phi_m(t') | \dot{\hat{H}}(t') | \phi_n(t') \rangle}{[E_n(t') - E_m(t')]} dt' \sim \frac{1}{Gt_f} \ll 1 \quad (4.1.104)$$

by the adiabatic criteria. Therefore, as long as the adiabatic criteria are satisfied, we never need to worry about time scales where non-adiabatic corrections start kicking in. This will be true for the leading order corrections. Subleading corrections still grow, but much more slowly as $\left|\frac{\dot{\phi}}{G}\right|, \left|\frac{\dot{\theta}}{G}\right| \ll 1$.

4.2. Changes to the initial state

In deriving (4.1.92), we assumed that the changes from the Schrieffer-Wolff transformation to the initial state are negligible. More specifically, applying the Schrieffer-Wolff transformation changes $\hat{\rho}(0) \rightarrow e^{\hat{S}(t)} \hat{\rho}(0) e^{-\hat{S}(t)}$ which will affect $\langle \tilde{\Pi}_{0d}(t) \rangle$ in a way that will be described in (4.2.17) below. In this section we give conditions for when we can ignore these changes. Recall that at initial times $t_i = 0$, the density matrix is given by

$$\hat{\rho}(0) = |\psi_s(0)\rangle \langle \psi_s(0)| \otimes \hat{\rho}_{env}(0), \quad (4.2.1)$$

where $\hat{\rho}_{env}(0)$ describes the cavity degrees of freedom being in a thermal state. The initial state of the system is given by

$$|\psi_s(0)\rangle = \alpha |0\rangle + \beta |g_1\rangle \quad (4.2.2)$$

Using (4.2.2) it is straightforward to see that

$$\hat{\rho}(0) = \left(|\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* |0\rangle \langle g_1| + \alpha^*\beta |g_1\rangle \langle 0| + |\beta|^2 |g_1\rangle \langle g_1| \right) \otimes \hat{\rho}_{env}(0) \quad (4.2.3)$$

Now, we want to see how the transformation $\hat{U}(t) = e^{\hat{S}(t)}$ affects the density matrix at initial times. This is important because if the changes to the density matrix at initial times arising from the Schrieffer-Wolff transformation cannot be neglected relative to $\hat{\rho}(0)$, then there would be terms in the Hamiltonian (other than $|d\rangle \langle d|$ in (4.1.91)) that would contribute to the phase of $\langle \tilde{\Pi}_{0d}(t) \rangle$.

Recall that we found that

$$\hat{S}(t) = \hat{\chi}_1(t) |+\rangle \langle d| + \hat{\chi}_2(t) |-\rangle \langle d| + \hat{\chi}_3(t) |+\rangle \langle -| + h.c. \quad (4.2.4)$$

with

$$\hat{\chi}_1(t) = \frac{1}{\sqrt{2}G} [h(t) - e^{-iGt}h(0)] - i\frac{g}{\sqrt{2}} \left[\frac{e^{i\Delta_1 t} \cos \theta(t) \hat{d}(t) - e^{-iGt} \hat{d}}{\Delta + G} - \frac{\sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \hat{d}(t)}{\Delta - G} \right], \quad (4.2.5)$$

and

$$\hat{\chi}_2(t) = -\frac{1}{\sqrt{2}G} [h(t) - e^{-iGt}h(0)] + i\frac{g}{\sqrt{2}} \left[\frac{e^{i\Delta_1 t} \cos \theta(t) \hat{d}(t) - e^{iGt} \hat{d}}{\Delta - G} - \frac{\sin \theta(t) e^{i\phi(t)} e^{i\Delta_2 t} \hat{d}(t)}{\Delta + G} \right]. \quad (4.2.6)$$

We don't need to specify $\hat{\chi}_3(t)$ since as we will see below it commutes with $\hat{\rho}(0)$. At this stage we need to evaluate

$$e^{\hat{S}(t)} \hat{\rho}(0) e^{-\hat{S}(t)} = \rho(0) + [\hat{S}(t), \hat{\rho}(0)] + \frac{1}{2} [\hat{S}(t), [\hat{S}(t), \hat{\rho}(0)]] + \dots \quad (4.2.7)$$

At time $t_i = 0$, the instantaneous eigenstates of the Hamiltonian are given by

$$|d\rangle = |g_1\rangle \quad (4.2.8)$$

$$|+\rangle = \frac{1}{\sqrt{2}} (+i|e\rangle - |g_2\rangle), \quad (4.2.9)$$

$$|-\rangle = -\frac{1}{\sqrt{2}} (i|e\rangle + |g_2\rangle). \quad (4.2.10)$$

Using (4.2.4) and (4.2.8) to (4.2.10), it is straightforward to show that

$$\begin{aligned} [\hat{S}(t), \hat{\rho}(0)] &= [\alpha^* \beta (\hat{\chi}_1(t) |+\rangle \langle 0| + \hat{\chi}_2(t) |-\rangle \langle 0|) \\ &\quad + |\beta|^2 (\hat{\chi}_1(t) |+\rangle \langle d| + \hat{\chi}_2(t) |-\rangle \langle d|)] \otimes \hat{\rho}_{env}(0) + h.c. \end{aligned} \quad (4.2.11)$$

Now the prefactor in (4.2.11) will scale as (using that $\hat{\chi}_1(t)$ scales the same way as $\hat{\chi}_2(t)$)

$$\hat{\chi}_1(t) \sim \max \left\{ \frac{\dot{\theta}}{G}, \frac{\dot{\phi}}{G}, \frac{g\sqrt{N_{th}}}{\Delta} \right\} \quad (4.2.12)$$

However, the observable that we are interested in calculating is $\langle \tilde{\Pi}_{0d}(t) \rangle$. We can use the results of (3.1.7) to (3.1.13) with the exception that the state $\hat{\rho}(0)$ has to be replaced with

$e^{\hat{S}(t)} \hat{\rho}(0) e^{-\hat{S}(t)}$ so that

$$\left\langle \tilde{\Pi}_{0d}(t) \right\rangle = \text{Tr} \left\{ e^{\hat{S}(t)} \hat{\rho}(0) e^{-\hat{S}(t)} e^{-i \int_0^t dt' L_v(t')} |0\rangle \langle d| \right\}, \quad (4.2.13)$$

where

$$L_v(t') |0\rangle \langle d| = \left[\tilde{H}'(t'), |0\rangle \langle d| \right]. \quad (4.2.14)$$

From (4.2.3), we will focus on the $\alpha^* \beta |d\rangle \langle 0|$ contribution to $\hat{\rho}(0)$ in (4.2.13) since we are only interested in calculating how the corrections to $\left\langle \tilde{\Pi}_{0d}(t) \right\rangle$ scale. Using the cyclic permutation property of the trace and the fact that the operation

$$e^{-i \int_0^t dt' L_v(t')} |0\rangle \langle d| = e^{-i \hat{\phi}_{0d}(t)} |0\rangle \langle d|, \quad (4.2.15)$$

with the operator $\hat{\phi}_{0d}(t)$ satisfying the property

$$\text{Tr} \left\{ e^{-i \hat{\phi}_{0d}(t)} \hat{\rho}_{env}(0) \right\} = \alpha \beta^* e^{-i \gamma_d(t)} \left\langle T_t e^{i \frac{g^2}{\Delta} \int_0^t \cos 2\theta(t') \hat{d}^\dagger(t') \hat{d}(t') dt'} \right\rangle, \quad (4.2.16)$$

which is what we found in (4.1.92). Now we can write (4.2.13) as

$$\left\langle \tilde{\Pi}_{0d}(t) \right\rangle = \text{Tr} \left\{ \langle d| e^{\hat{S}(t)} |d\rangle e^{-i \hat{\phi}_{0d}(t)} \hat{\rho}_{env}(0) \right\} \quad (4.2.17)$$

where we used the fact that $e^{\hat{S}(t)} |0\rangle = |0\rangle$. The prefactor $\langle d| e^{\hat{S}(t)} |d\rangle$ in (4.2.17) will not cause decay to zero in the chosen parameter regime. To understand this, we note that the exponential decay arising from $\hat{\phi}_{0d}(t)$ is due to the fact that we integrate over terms proportional to $\hat{d}^\dagger(t) \hat{d}(t)$. However, $\hat{S}(t)$ only contains terms proportional to $\hat{d}(t)$ (from $\hat{\chi}_{1,2}(t)$) which are not being integrated over in $\langle d| e^{\hat{S}(t)} |d\rangle$. Thus the term $\langle d| e^{\hat{S}(t)} |d\rangle$ will not give rise to exponential decay for the chosen parameter regime.

From (4.2.4), a quick calculation shows that

$$\begin{aligned}
\langle d | e^{\hat{S}(t)} | d \rangle &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \langle d | \hat{S}^{2k} | d \rangle \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\hat{\chi}_1^\dagger(t) \hat{\chi}_1(t) + \hat{\chi}_2^\dagger(t) \hat{\chi}_2(t) \right)^k \\
&= \cos \sqrt{\hat{\chi}_1^\dagger(t) \hat{\chi}_1(t) + \hat{\chi}_2^\dagger(t) \hat{\chi}_2(t)} \\
&= 1 - \frac{1}{2} \left(\hat{\chi}_1^\dagger(t) \hat{\chi}_1(t) + \hat{\chi}_2^\dagger(t) \hat{\chi}_2(t) \right) + \mathcal{O}(\hat{S}(t)^4)
\end{aligned} \tag{4.2.18}$$

Going back to the definitions of $\hat{\chi}_1(t)$ and $\hat{\chi}_2(t)$ given in (4.2.5) and (4.2.6), we can show that

$$\left\| \hat{\chi}_1^\dagger(t) \hat{\chi}_1(t) + \hat{\chi}_2^\dagger(t) \hat{\chi}_2(t) \right\| \sim \text{Max} \left\{ \frac{1}{t_f^2 G^2}, \frac{g \sqrt{N_{th}}}{G t_f \Delta}, \frac{g^2 N_{th}}{\Delta^2} \right\} \tag{4.2.19}$$

In the next section, we will see that on short time scales the fidelity that we obtain (which will not include the corrections in (4.2.19)) will be very close to unity. If we included the leading order correction terms in (4.2.18), these terms would create small oscillations about the reported fidelity. Ergo, when we report an error rate ϵ at a time t_f due to dephasing effects, we need to ensure that the correction terms in (4.2.19) are much smaller than ϵ .

4.3. Calculation of the fidelity for the case where $g_1 = g_2$

The goal of this section will be to obtain the fidelity for our state transfer protocol when the two cavity coupling constants are identical. It is important to remember that (4.1.92) is only valid when $g_1 = g_2$. To keep the analysis simple, the state will be transferred using a path such that $\theta(t)$ is a linear function of time. Consequently, the path in $\{\Omega_1, \Omega_2\}$ space will be described by (3.2.18) and (3.2.19) with $\theta(t) = -bt$ where

$$b \equiv \frac{2n\pi}{t_f}, \tag{4.3.1}$$

and n is a parameter that controls how many loops will be done during the evolution. To calculate the fidelity, we must first calculate the correlation function found in (4.1.92) at

the final time of the state transfer protocol. We start by defining

$$\hat{X}(t) \equiv \frac{g^2}{\Delta} \int_0^t dt' \cos\left(\frac{4\pi n t'}{t_f}\right) \hat{d}^\dagger(t') \hat{d}(t'). \quad (4.3.2)$$

From this definition we can rewrite (4.1.92) when $t = t_f$ as

$$\left\langle \hat{\Pi}_{0d}(t_f) \right\rangle = \alpha \beta^* e^{-i\gamma_d(t)} \left\langle T_t e^{i\hat{X}(t_f)} \right\rangle. \quad (4.3.3)$$

The quantum noise contribution to the cavity lowering operator for a cavity driven by a single classical laser field was given in Eq. (2.4.31). When we performed the displacement transformation in (4.1.6), only the cavity frequency appears in the time dependent exponential. This is because we are working in an interaction picture at the cavity resonance frequency. With this particular type of transformation we have that

$$\hat{d}(t) = -\sqrt{\kappa} \int_{-\infty}^t e^{-\frac{\kappa}{2}(t-\tau)} \hat{\xi}(\tau) d\tau. \quad (4.3.4)$$

We remind the reader that κ represents the cavity damping rate and $\hat{\xi}(t)$ describes both thermal and vacuum noise incident on the cavity through the drive port. Also, it has the auto-correlation function given by

$$\left\langle \hat{\xi}^\dagger(t) \hat{\xi}(t') \right\rangle = N_{th} \delta(t - t'), \quad (4.3.5)$$

where N_{th} is the bosonic thermal equilibrium occupation number. Given these results, a straightforward calculation shows that

$$\left\langle \hat{d}^\dagger(t_1) \hat{d}(t_2) \right\rangle = N_{th} e^{-\frac{\kappa}{2}|t_1 - t_2|}. \quad (4.3.6)$$

The rest of this section will be devoted to calculating $\left\langle T_t e^{-i\hat{X}(t_f)} \right\rangle$ and finding the fidelity as a function of t_f for the state transfer protocol. We will expand the exponential in its Taylor series and perform a moment expansion allowing us to keep only the second order term ($\left\langle T_t \hat{X}(t_f) \hat{X}(t_f) \right\rangle$). This can be justified by choosing the right conditions on our parameters such that higher order terms will give rise to much smaller contributions. These conditions will be established below.

The leading-order term in the moment expansion will involve the correlation function

$$\langle \hat{X}(t_f) \rangle = -\frac{g^2}{\Delta} \int_0^{t_f} dt \cos\left(\frac{4n\pi t}{t_f}\right) \langle \hat{d}^\dagger(t) \hat{d}(t) \rangle. \quad (4.3.7)$$

From (4.3.6), we see that

$$\langle \hat{X}(t_f) \rangle = -\frac{N_{th}g^2}{\Delta} \int_0^{t_f} dt \cos\left(\frac{4n\pi t}{t_f}\right) = 0. \quad (4.3.8)$$

Consequently, the leading order term vanishes. The next order term (and also the most important one) is given by

$$\langle T_t \hat{X}(t_f) \hat{X}(t_f) \rangle = \left(-\frac{g^2}{\Delta}\right)^2 \int_0^{t_f} dt_1 dt_2 \cos\left(\frac{4\pi n t_1}{t_f}\right) \cos\left(\frac{4\pi n t_2}{t_f}\right) \langle T_t \hat{d}^\dagger(t_1) \hat{d}(t_1) \hat{d}^\dagger(t_2) \hat{d}(t_2) \rangle. \quad (4.3.9)$$

We can use Wick's theorem to calculate the correlation function appearing in the above expression. However, since there are integral factors appearing in Eq. (4.3.9), we can simplify much of the notation by defining the operator

$$L^{(n)}(t_f) \mathcal{A} \equiv \int_0^{t_f} \prod_{i=1}^n dt_i \cos\left(\frac{4\pi t_i}{t_f}\right) \mathcal{A}. \quad (4.3.10)$$

Then using this definition and Wick's theorem we find

$$\begin{aligned} L^{(2)}(t_f) \langle T_t \hat{d}^\dagger(t_1) \hat{d}(t_1) \hat{d}^\dagger(t_2) \hat{d}(t_2) \rangle &= L^{(2)}(t_f) \left\{ \langle \hat{d}^\dagger(t_1) \hat{d}(t_2) \rangle \langle \hat{d}(t_1) \hat{d}^\dagger(t_2) \rangle \Theta(t_1 - t_2) \right. \\ &\quad \left. + \langle \hat{d}^\dagger(t_2) \hat{d}(t_1) \rangle \langle \hat{d}(t_2) \hat{d}^\dagger(t_1) \rangle \Theta(t_2 - t_1) \right\}. \end{aligned} \quad (4.3.11)$$

Note that correlation functions of the \hat{d} operator evaluated at equal times will always vanish when evaluated under the integral (just like what we showed in (4.3.8)). Using (4.3.6) along with (2.4.35), the product of correlation functions appearing in the above equation is found to be

$$\langle \hat{d}^\dagger(t_1) \hat{d}(t_2) \rangle \langle \hat{d}(t_1) \hat{d}^\dagger(t_2) \rangle = N_{th}(N_{th} + 1) e^{-\kappa|t_1 - t_2|}. \quad (4.3.12)$$

Putting everything together we find that

$$\left\langle T_t \hat{X}(t_f) \hat{X}(t_f) \right\rangle = \left(\frac{g^2}{\Delta} \right)^2 N_{th} (N_{th} + 1) \int_0^{t_f} dt_1 dt_2 \cos \left(\frac{4\pi n t_1}{t_f} \right) \cos \left(\frac{4\pi n t_2}{t_f} \right) e^{-\kappa |t_1 - t_2|}. \quad (4.3.13)$$

This integral is straightforward to compute and yields one of the central results of this thesis

$$\left\langle T_t \hat{X}(t_f) \hat{X}(t_f) \right\rangle = N_{th} (N_{th} + 1) \left(\frac{g^2}{\Delta} \right)^2 \frac{\kappa t_f^3 \left[4n^2 \pi^2 + 2\kappa t_f (e^{-\kappa t_f} - 1) + \kappa^2 t_f^2 \right]}{\left(4n^2 \pi^2 + \kappa^2 t_f^2 \right)^2}. \quad (4.3.14)$$

We will come back to the implications of this expression shortly. For now, we will focus on determining how the third-order term scales in terms of the relevant parameters. This can be determined from

$$\left\langle T_t \hat{X}(t_f) \hat{X}(t_f) \hat{X}(t_f) \right\rangle = \left(-\frac{g^2}{\Delta} \right)^3 L^{(3)}(t_f) \left\langle T_t \hat{d}^\dagger(t_1) \hat{d}(t_1) \hat{d}^\dagger(t_2) \hat{d}(t_2) \hat{d}^\dagger(t_3) \hat{d}(t_3) \right\rangle. \quad (4.3.15)$$

When expanding the term on the right hand side of the above equation, there will be $3! = 6$ terms coming from the time ordering symbol. We can evaluate one of them to determine how they scale with respect to the relevant parameters. As an example, we pick

$$J_1 \equiv L^{(3)}(t_f) \left\langle \hat{d}^\dagger(t_1) \hat{d}(t_1) \hat{d}^\dagger(t_2) \hat{d}(t_2) \hat{d}^\dagger(t_3) \hat{d}(t_3) \right\rangle \Theta(t_1 - t_2) \Theta(t_2 - t_3). \quad (4.3.16)$$

Expanding using Wick's theorem we get

$$\begin{aligned} J_1 = & L^{(3)}(t_f) \left\langle \hat{d}^\dagger(t_1) \hat{d}(t_2) \right\rangle \left\langle \hat{d}(t_1) \hat{d}^\dagger(t_3) \right\rangle \left\langle \hat{d}^\dagger(t_2) \hat{d}(t_3) \right\rangle \Theta(t_1 - t_2) \Theta(t_2 - t_3) \\ & + L^{(3)}(t_f) \left\langle \hat{d}^\dagger(t_1) \hat{d}(t_3) \right\rangle \left\langle \hat{d}(t_1) \hat{d}^\dagger(t_2) \right\rangle \left\langle \hat{d}(t_2) \hat{d}^\dagger(t_3) \right\rangle \Theta(t_1 - t_2) \Theta(t_2 - t_3). \end{aligned} \quad (4.3.17)$$

All the other terms will vanish since at least one correlation function would be evaluated at equal times thus vanishing under the action of the integral. We can evaluate the products using (4.3.6) and find that

$$\left\langle \hat{d}^\dagger(t_1) \hat{d}(t_2) \right\rangle \left\langle \hat{d}(t_1) \hat{d}^\dagger(t_3) \right\rangle \left\langle \hat{d}^\dagger(t_2) \hat{d}(t_3) \right\rangle = N_{th}^2 (N_{th} + 1) e^{-\frac{\kappa}{2}(|t_1 - t_2| + |t_1 - t_3| + |t_2 - t_3|)}. \quad (4.3.18)$$

Evaluating this term under the integral, we find that (for $n = 1$)

$$N_{th}^2 (N_{th} + 1) L^{(3)}(t_f) e^{-\frac{\kappa}{2}(|t_1 - t_2| + |t_1 - t_3| + |t_2 - t_3|)} \Theta(t_1 - t_2) \Theta(t_2 - t_3) = -\frac{2N_{th}^2 (N_{th} + 1) (1 - e^{-\kappa t_f}) \kappa t_f^4}{1024\pi^4 + 80\pi^2 \kappa^2 t_f^2 + \kappa^4 t_f^4}. \quad (4.3.19)$$

We can now determine how the third-order contribution scales relative to equation (4.3.14).

We first begin by considering the long time limit, where $\kappa t_f \gg 1$. Then the second-order term will scale as

$$\langle T_t \hat{X}(t_f) \hat{X}(t_f) \rangle \sim \frac{N_{th} (N_{th} + 1) g^4 t_f}{\Delta^2 \kappa}. \quad (4.3.20)$$

The third-order term is seen to scale as

$$\langle T_t \hat{X}(t_f) \hat{X}(t_f) \hat{X}(t_f) \rangle \sim \frac{N_{th}^2 (N_{th} + 1) g^6}{\Delta^3} \frac{1}{\kappa^3}. \quad (4.3.21)$$

Consequently, to keep only leading order terms in the large time limit, we require

$$\frac{N_{th} g^2}{\Delta \kappa^2 t_f} \ll 1. \quad (4.3.22)$$

In the opposite (short-time) limit, $\kappa t_f \ll 1$, it can be shown that the leading-order term will scale as

$$\langle T_t \hat{X}(t_f) \hat{X}(t_f) \rangle \sim \frac{N_{th} (N_{th} + 1) g^4 \kappa n^2 t_f^3}{\Delta^2}. \quad (4.3.23)$$

whereas the third-order term scales as

$$\langle T_t \hat{X}(t_f) \hat{X}(t_f) \hat{X}(t_f) \rangle \sim \frac{N_{th}^2 (N_{th} + 1) g^6 \kappa^2 n^3 t_f^5}{\Delta^3}. \quad (4.3.24)$$

To keep only leading-order terms in this limit, we require that

$$\frac{N_{th} g^2 \kappa t_f^2 n}{\Delta} \ll 1. \quad (4.3.25)$$

Assuming that conditions (4.3.22) and (4.3.25) are satisfied, we can perform a moment expansion in evaluating $\langle T_t e^{i\hat{X}} \rangle$. To do so we use the following identity

$$\langle T_t e^{-i\hat{X}} \rangle = e^{\ln \langle T_t e^{-i\hat{X}} \rangle}. \quad (4.3.26)$$

Expanding we have that

$$\ln \langle T_t e^{-i\hat{X}} \rangle = \ln \left[1 - \left(\frac{1}{2} \langle T_t \hat{X}^2 \rangle - \frac{i}{6} \langle T_t \hat{X}^3 \rangle + \mathcal{O}(\langle \hat{X}^4 \rangle) \right) \right]. \quad (4.3.27)$$

Expanding the logarithm, it can be approximated by

$$\ln \langle T_t e^{i\hat{X}} \rangle \approx -\frac{1}{2} \langle T_t \hat{X}^2 \rangle - \frac{1}{8} \langle T_t \hat{X}^2 \rangle^2 + \frac{i}{6} \langle T_t \hat{X}^3 \rangle. \quad (4.3.28)$$

Thus, if (4.3.22) is satisfied we can write

$$\langle T_t e^{i\hat{X}} \rangle \approx e^{-\frac{1}{2} \langle T_t \hat{X}(t_f) \hat{X}(t_f) \rangle}. \quad (4.3.29)$$

This means that under this approximation, we only need to compute the correlation function $\langle \hat{T}_t \hat{X}(t_f) \hat{X}(t_f) \rangle$ to obtain the correction to the geometric phase arising from the fluctuating number of photons inside the cavity. Using (4.3.14) and (3.2.1), the average of the relevant component of the density matrix for our state transfer protocol is found to be

$$\langle \tilde{\Pi}_{0d}(t_f) \rangle = \alpha \beta^* e^{i\gamma_d(t)} \exp \left[-\frac{N_{th}(N_{th}+1)}{2} \left(\frac{g^2}{\Delta} \right)^2 \frac{\kappa t_f^3 [4n^2\pi^2 + 2\kappa t_f (e^{-\kappa t_f} - 1) + \kappa^2 t_f^2]}{(4n^2\pi^2 + \kappa^2 t_f^2)^2} \right]. \quad (4.3.30)$$

From this result, the fidelity is found to be given by

$$F = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \cos(\gamma_d(t_f)) \exp \left(-\frac{N_{th}(N_{th}+1)}{2} \left(\frac{g^2}{\Delta} \right)^2 \frac{\kappa t_f^3 [4n^2\pi^2 + 2\kappa t_f (e^{-\kappa t_f} - 1) + \kappa^2 t_f^2]}{(4n^2\pi^2 + \kappa^2 t_f^2)^2} \right), \quad (4.3.31)$$

where we remind the reader that $\gamma_d(t)$ was found to be

$$\gamma_d(t) = \int_0^t \dot{\phi}(t') \sin^2 \theta(t') dt'. \quad (4.3.32)$$

To get more intuition on the behavior of the fidelity, we will consider certain limits. If we first consider the limit where $\kappa t_f \gg 1$ and $n = 1$, the fidelity will reduce to

$$F \approx |\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \cos(\gamma_d(t_f)) \exp \left(-\frac{N_{th}(N_{th}+1)}{2} \left(\frac{g^2}{\Delta} \right)^2 \frac{t_f}{\kappa} \right), \quad (4.3.33)$$

and so the exponent depends linearly on the state transfer time. Recall that in the $\kappa t_f \gg 1$ limit, the condition to neglect the third moment was $\frac{N_{th} g^2}{\Delta \kappa^2 t_f} \ll 1$. Since the exponential in (4.3.33) scales as $\frac{N_{th}^2 g^4 t_f}{\Delta^2 \kappa}$, it can still be made large as long as $\Delta^3 \kappa^3 \sim N_{th}^3 g^6$.

In the opposite limit where $\kappa t_f \ll 1$, the fidelity is approximately given by

$$F \approx |\alpha|^4 + |\beta|^4 + 2 |\alpha|^2 |\beta|^2 \cos(\gamma_d(t_f)) \exp\left(-\frac{N_{th}(N_{th}+1)}{2} \left(\frac{g^2}{\Delta}\right)^2 \frac{\kappa t_f^3}{16\pi^2}\right). \quad (4.3.34)$$

Coming back to the condition $\frac{N_{th} g^2 \kappa t_f^2}{\Delta} \ll 1$ that allowed us to neglect the third moment, the exponent in (4.3.34) scales as $\frac{N_{th}^2 g^4 \kappa t_f^3}{\Delta^2}$. Since neglecting the third moment requires having $\frac{N_{th} g^2 t_f}{\Delta} \lesssim 1$, we conclude that the terms in the exponent will always be much smaller than 1. In the short-time regime, (4.3.34) shows that the decay is super-exponential ($\sim e^{-(t_f/\tau)^3}$). This is analogous to the case of a Hahn-echo decay with a Lorentzian spectral density since in the short-time limit the behavior is similar (apart for constant prefactors). The reader is referred to [45] for more details.

In the next section we will propose a scheme that allows us to improve the fidelity by performing the state transfer sequence over many cycles, directly analogous to an n-pulse Carr-Purcell-Meiboom-Gill (CPMG) sequence (see section (4.4) for more details). Consequently, it would be useful to compare the fidelity arising from a many cycle evolution to the case where we left the system in its initial superposition state $|\psi(0)\rangle = \alpha |0\rangle + \beta |g_1\rangle$ (this would be equivalent to setting the function $\cos 2\theta(t) = 1$). In other words, if we prepared our qubit state in the linear combination $\alpha |0\rangle + \beta |g_1\rangle$, the system would not pick up a geometric phase (since there would be no lasers creating transition between the different levels and so there would be no closed loop evolution) but it would still decohere due to the noise inside the cavity. We assume that even though the cavity isn't being driven (so as to leave the qubit state in the superposition $\alpha |0\rangle + \beta |g_1\rangle$), there is still noise inside the cavity arising from fluctuations in the number of photons. Consequently, our scheme would be extremely useful since it would offer a way of prolonging the life of the qubit. Of course, one must keep in mind that the ultimate goal is to perform a phase gate and to do this the state $\alpha |0\rangle + \beta |g_1\rangle$ needs to pick a geometric phase. So the whole idea behind our state

transfer protocol is to minimize the effects of dephasing with the intention of performing a phase gate.

To determine the dephasing effects from keeping the qubit state in its initial state, we only have to consider a Hamiltonian (in an interaction picture) with the noise terms

$$\hat{H}(t) = \delta\hat{\omega}_1 |g_1\rangle \langle g_1| + \delta\hat{\omega}_2 |g_2\rangle \langle g_2| + \hat{H}_{env}(\delta\hat{\omega}_1, \delta\hat{\omega}_2). \quad (4.3.35)$$

Since the above Hamiltonian is purely diagonal, we don't need to write it in terms of a superadiabatic basis. Instead we can work in the original basis and compute the average of the operator

$$\tilde{\Pi}_{0g_1} \equiv |0\rangle \langle g_1|. \quad (4.3.36)$$

Note however that $|g_1\rangle = |d(t_i)\rangle = |d\rangle$ so that $\tilde{\Pi}_{0g_1} = \tilde{\Pi}_{0d}$. We can apply the same procedure as we did in equations (3.1.6) to (3.1.18). Doing so we find

$$\langle \tilde{\Pi}_{0g_1}(t) \rangle = \langle \tilde{\Pi}_{0g_1}(0) \rangle \langle T_t e^{-i \int_0^t dt' \delta\hat{\omega}_1(t')} \rangle. \quad (4.3.37)$$

where we set $g_1 = g_2$ so that $\delta\hat{\omega}_1 = -\delta\hat{\omega}_2$ and $\delta\hat{\omega}_1 = \frac{g^2}{\Delta} \hat{d}^\dagger \hat{d}$. If we perform a moment expansion on the term $\langle T_t e^{-i \int_0^t dt' \delta\hat{\omega}_1(t')} \rangle$ and keep the second order term as we did previously, and using (4.2.3), the fidelity is found to be

$$F = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \exp \left(-\frac{N_{th}(N_{th} + 1)}{\kappa^2} \left(\frac{g^2}{\Delta} \right)^2 (e^{-\kappa t_f} - 1 + \kappa t_f) \right). \quad (4.3.38)$$

Before comparing the fidelity obtained for our state-transfer protocol to the one obtained by keeping the qubit in its original state, we will consider certain limits of the above expression. If we first consider the limit where $\kappa t_f \gg 1$, the fidelity will reduce to

$$F \approx |\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \exp \left(-\frac{N_{th}(N_{th} + 1)}{\kappa} \left(\frac{g^2}{\Delta} \right)^2 t_f \right). \quad (4.3.39)$$

Just as in the state-transfer case, the long-time limit corresponds to a linear dependence on time (total state transfer time) for the exponent. Notice the factor of $\frac{1}{2}$ appearing in (4.3.33) not present in (4.3.39). This can be understood from the fact that if the state remains in $|g_1\rangle$ for the entire state transfer protocol, it will only see the noise from $\delta\hat{\omega}_1$. However, for

our state transfer protocol, half of the time the state will see as much of $|g_1\rangle$ and $|g_2\rangle$. As we mentioned in section (3.2), the average of the noise Hamiltonian in the state $|g_1\rangle + |g_2\rangle$ is $\delta\hat{\omega}_1 + \delta\hat{\omega}_2 = 0$. Consequently, we expect the fidelity to improve by a factor of 2 relative to the case where we stay in $|g_1\rangle$ for the entire state transfer.

In the opposite limit where $\kappa t_f \ll 1$, the fidelity is approximately given by

$$F \approx |\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 \exp\left(-\frac{N_{th}(N_{th}+1)}{2} \left(\frac{g^2}{\Delta}\right)^2 t_f^2\right). \quad (4.3.40)$$

Contrary to the state transfer case, in the short-time limit the exponent depends on t_f^2 instead of t_f^3 and also has no dependence on the damping rate.

4.4. Many-cycle evolution

Careful observation of Eq. (4.3.31) shows that in the short time limit the exponent scales as $1/n^2$ where the integer n corresponds to the number of state-transfer sequences performed in parameter space. It is thus seen that as n gets larger the exponential will become closer to unity giving rise to a better fidelity. This suggests that it is very advantageous to perform the state-transfer protocol over many cycles, especially for state preservation. The reason being that having a longer coherence time (arising from transferring information back and fourth) would allow one to store information for longer time scales. The closed path in parameter space effectively averages the noise over progressively shorter time scales as n is increased, similar to an n -pulse CPMG sequence, for which $s(t) = \cos 2\theta(t)$ alternates between ± 1 .

To trust the results derived in this chapter, one must be careful to obey the adiabatic criteria (which could fail if n is chosen to be large enough). We know that one of the relevant conditions that needs to be satisfied is $\frac{\dot{\theta}}{G} \ll 1$. For a circular state transfer in parameter space, this condition reduces to

$$\frac{n\pi}{Gt_f} \ll 1. \quad (4.4.1)$$

Thus, as long as G is chosen to satisfy the condition (4.4.1) and the short-time condition is satisfied, we can safely apply our adiabatic theory developed above for the quantum state transfer protocol over many evolution cycles. Using (4.3.31) and (4.3.38) we can plot the

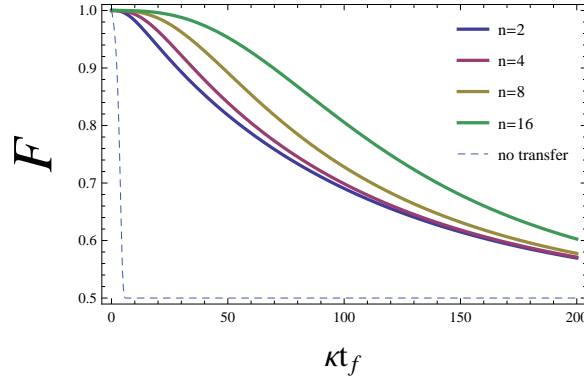


FIGURE 4.4.1. Many-cycle evolution

Plot of the fidelity as a function of t_f (the total state transfer time). We have chosen the values $\alpha = \beta = \frac{1}{2}$, $N_{th} = 1$, $\frac{g^2}{\Delta} = 0.1$ and $\kappa = 1$. However, each curve corresponds to a particular value of the state transfer cycles (n) illustrated in the plot legend. The dashed curve corresponds to the fidelity that one would get if no state transfer was performed. Note that we set the phase $\phi(t) = 0$ so that the closed system Berry's phase ($\gamma_d(t_f)$) vanishes. Note that the chosen values are consistent with the conditions (4.3.22) and (4.3.25) allowing us to neglect the third moment in section (4.3).

fidelity as a function of the total state transfer time allowing us to compare the improvement due to a many-cycle evolution over leaving the state in its initial superposition.

Figure (4.4.1) clearly shows a significant advantage in performing the state-transfer sequence compared to leaving the qubit in its initial state. Furthermore, performing the state-transfer protocol over many sequences allows one to fight decoherence even further. Consequently, to preserve a quantum state for longer time periods before all the information is lost due to decoherence effects, one can perform a state transfer sequence by driving the cavity with two laser fields which will induce transitions between the states of the two atoms inside the cavity. To preserve the qubit state of atom 1 on a longer time scale, one can keep driving the cavity over many cycles which will improve its coherence time. Of course, we remind the reader that we are limited by the adiabatic regime in the number of times one can perform the state transfer sequence.

We end this section by considering a many-cycle evolution of the four-level system using experimental values for the relevant physical parameters. Following [49], we take the value for the cavity coupling constant to be

$$g = 50 \times 10^3 Hz, \quad (4.4.2)$$

the cavity damping rate is chosen to be

$$\kappa = 10^5 Hz. \quad (4.4.3)$$

To ensure that the conditions in table (4.1.2) are satisfied, we choose the detuning frequency to be given by

$$\Delta = 10^6 Hz, \quad (4.4.4)$$

the thermal occupation number

$$N_{th} = 1, \quad (4.4.5)$$

and G to be

$$G = 10^5 Hz. \quad (4.4.6)$$

In plotting the fidelity, we use the expression obtained in (4.3.31) rewritten using the dimensionless parameter κt_f :

$$F = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \left\{ \cos(\gamma_d(t_f)) \exp \left(-\frac{N_{th}(N_{th}+1)}{2} \left(\frac{g^2}{\Delta} \right)^2 \frac{(\kappa t_f)^3 [4n^2\pi^2 + 2\kappa t_f(e^{-\kappa t_f} - 1) + (\kappa t_f)^2]}{\kappa^2 (4n^2\pi^2 + (\kappa t_f)^2)^2} \right) \right\} \quad (4.4.7)$$

and for the case where we don't perform the state transfer, the expression for the fidelity was found in (4.3.38) to be given by

$$F = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \exp \left(-\frac{N_{th}(N_{th}+1)}{\kappa^2} \left(\frac{g^2}{\Delta} \right)^2 (e^{-\kappa t_f} - 1 + \kappa t_f) \right). \quad (4.4.8)$$

As expected, figure (4.4.2) shows that as we increase the number of state-transfer cycles (n), the fidelity remains closer to unity for longer time scales. Also, the dashed curve

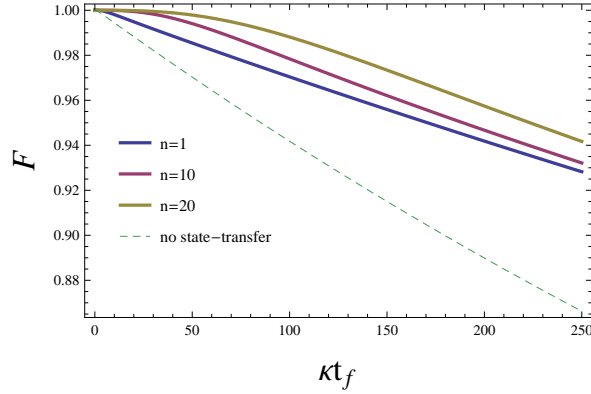


FIGURE 4.4.2. Fidelity using experimental values for the system parameters
Plot of the fidelity as a function of t_f (the total state transfer time) using the experimental values of (4.4.2) to (4.4.5). We set the parameters $\alpha = \beta = \frac{1}{2}$. However, each curve corresponds to a particular value of the state transfer cycles (n) illustrated in the plot legend. The dashed curve corresponds to the fidelity that one would get if no state transfer was performed. Note that we set the phase $\phi(t) = 0$ so that the closed system Berry's phase ($\gamma_d(t_f)$) vanishes.

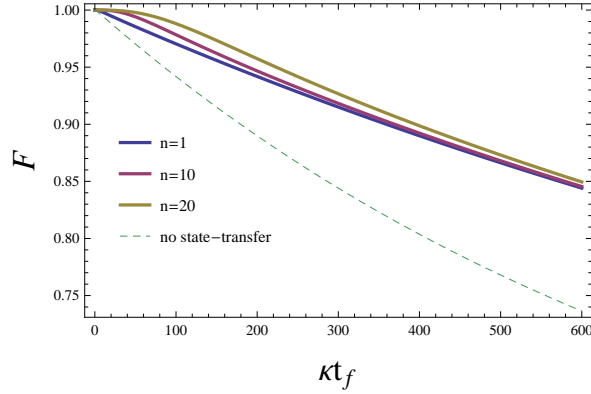


FIGURE 4.4.3. Fidelity using experimental values for the system parameters with extended state-transfer time
Plot of the fidelity as a function of t_f (the total state transfer time) using the experimental values of (4.4.2) to (4.4.5). We set the parameters $\alpha = \beta = \frac{1}{2}$. We set the phase $\phi(t) = 0$ so that the closed system Berry's phase ($\gamma_d(t_f)$) vanishes. We see that for longer times, the improvement of the fidelity by increasing the number of state transfer cycles decreases.

representing dephasing effects for the case where no state-transfer is performed decays much more quickly than the state-transfer protocol curves (thick curves). Consequently, we see that our state-transfer protocol offers a significant advantage for fighting dephasing effects when using experimental values for the parameters of interest.

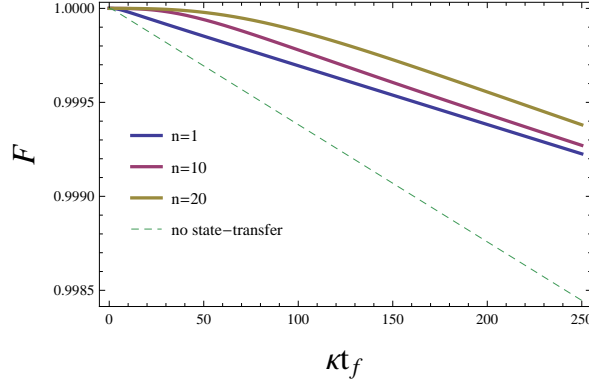


FIGURE 4.4.4. Fidelity using experimental values for the system parameters with increased detuning frequency

Plot of the fidelity as a function of t_f (the total state transfer time) using the experimental values of (4.4.2) to (4.4.5). We set the parameters $\alpha = \beta = \frac{1}{2}$. We set the phase $\phi(t) = 0$ so that the closed system Berry's phase ($\gamma_d(t_f)$) vanishes. We increased the value of the detuning frequency to $\Delta = 10^7 \text{ Hz}$. It is thus observed that for larger detuning frequencies, the fidelity remains closer to unity for longer time scales.

4.5. Short-time Expansion

Upon careful consideration of the short time limit of the fidelity for our state transfer protocol (Eq. (4.3.34)), the t_f^3 dependence of the exponent is analogous to the case of a Carr-Purcell sequence for coherence control [45]. This scheme uses a sequence of Pi-pulses to fight decoherence (extend the coherence time of the qubit). In the short-time limit, the average of the spin 1/2 matrix $\langle \sigma_+ \rangle$ (also known as spin-coherence) also exhibits a super-exponential $\sim e^{-(t_f/\tau)^3}$ decay. However, the CPMG cycle uses a sequence of evenly spaced Pi-pulses in order to extend the life of the qubit [46]. To further extend the life of the qubit, an optimal sequence of Pi-pulses was proposed by G.S. Uhrig [47]. Based on these findings, we propose a scheme for finding an optimal path to perform our state transfer protocol via a short time expansion.

The first step is to perform a short time expansion on $C(t_f) \equiv \langle T_t e^{i\hat{X}(t_f)} \rangle$. We remind the reader that

$$\hat{X}(t) = \int_0^t dt' \cos(2\theta(t', t_f)) \delta\hat{\omega}_1(t'), \quad (4.5.1)$$

where

$$\delta\hat{\omega}_1 = \frac{g^2}{\Delta} \hat{d}^\dagger \hat{d}. \quad (4.5.2)$$

Notice that θ is also a function of t_f which is reflected in (4.5.1). For example, if we had a circular evolution, we know that $\cos(2\theta(t, t_f)) = \cos\left(\frac{4n\pi t}{t_f}\right)$. For simplicity let us also define the function

$$s(t, t_f) \equiv \cos(2\theta(t, t_f)). \quad (4.5.3)$$

Also, for a purely symmetric evolution the mean $\langle \hat{X}(t_f) \rangle = 0$. Performing the same set of approximations as we did in previous sections, we can write $C(t_f) \approx e^{-\frac{1}{2}\langle T_t \hat{X}^2(t_f) \rangle}$ where we have that

$$\langle T_t \hat{X}^2(t_f) \rangle = \left(\frac{g^2}{\Delta}\right)^2 \int_0^{t_f} dt_1 \int_0^{t_f} dt_2 s(t_1, t_f) s(t_2, t_f) \langle \hat{d}^\dagger(t_1) \hat{d}(t_1) \hat{d}^\dagger(t_2) \hat{d}(t_2) \rangle. \quad (4.5.4)$$

Notice that we are keeping things completely general since we have not specified a function for $s(t, t_f)$. Let us define

$$S_{\delta\omega}(t_1 - t_2) \equiv \left(\frac{g^2}{\Delta}\right)^2 \langle \hat{d}^\dagger(t_1) \hat{d}(t_1) \hat{d}^\dagger(t_2) \hat{d}(t_2) \rangle. \quad (4.5.5)$$

We can introduce its Fourier transform $S_{\delta\omega}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} S_{\delta\omega}(\omega)$ into equation (4.5.4) and rewrite it as

$$\langle T_t \hat{X}^2(t_f) \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{\delta\omega}(\omega) \int_0^{t_f} dt_1 e^{i\omega t_1} s(t_1, t_f) \int_0^{t_f} dt_2 e^{-i\omega t_2} s(t_2, t_f). \quad (4.5.6)$$

Now let us define

$$f(\omega, t_f) = \int_0^{t_f} dt e^{i\omega t} s(t, t_f). \quad (4.5.7)$$

With this definition and defining the filter function $\mathcal{F}(\omega, t_f) \equiv |f(\omega, t_f)|^2$, we can rewrite equation (4.5.6) as

$$\langle T_t \hat{X}^2(t_f) \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{\delta\omega}(\omega) \mathcal{F}(\omega, t_f). \quad (4.5.8)$$

We now use a similar procedure to what was done in [47] in order to find an optimized sequence of Pi-pulses. Instead of choosing the function $\cos(2\theta(t, t_f)) = \cos\left(\frac{2\pi n t}{t_f}\right)$ which corresponds to an evolution over n cycles, we will represent $\cos(2\theta(t, t_f))$ by a Fourier series

with a finite frequency cutoff given by

$$\cos [2\theta (t, t_f)] = \sum_{m=1}^n \left(A_m \cos \left(\frac{2\pi m t}{t_f} \right) + B_m \sin \left(\frac{2\pi m t}{t_f} \right) \right). \quad (4.5.9)$$

Using (4.5.7) and (4.5.9), we find that

$$\bar{f}(z) = z \sum_{m=1}^n \frac{(1 - e^{iz}) (iz A_m - 2\pi m B_m)}{z^2 - 4\pi^2 m^2}. \quad (4.5.10)$$

where $z = \omega t_f$. In terms of the function $\bar{f}(z)$, we can rewrite (4.5.8) as

$$\left\langle T_{\tau} \hat{X}^2(t_f) \right\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{S_{\delta_{\omega}}(\omega)}{\omega^2} |\bar{f}(\omega t_f)|^2. \quad (4.5.11)$$

Note that the coefficients A_m and B_m must be chosen such that they ensure that (4.5.9) is bounded between ± 1 . For now, we will ignore this and focus on determining them to make the first n derivatives of (4.5.10) vanish (see the procedure below). In order to have the fidelity as close to unity as possible for short time scales, we perform a short-time expansion by setting as many derivatives of (4.5.10) to zero (when $z = 0$). We give here the first few results:

$$\bar{f}^{(1)}(z)|_{z=0} = 0, \quad (4.5.12)$$

$$\bar{f}^{(2)}(z)|_{z=0} = -\frac{1}{\pi} \sum_{m=1}^n \frac{B_m}{m}, \quad (4.5.13)$$

$$\bar{f}^{(3)}(z)|_{z=0} = -\frac{3}{2\pi^2} \sum_{m=1}^n \left(\frac{A_m}{m^2} - \frac{B_m}{m} \right), \quad (4.5.14)$$

$$\bar{f}^{(4)}(z)|_{z=0} = \frac{i}{\pi^3} \sum_{m=1}^n \left(-3\pi \frac{A_m}{m^2} - 3\frac{B_m}{m^3} + 2\pi^2 \frac{B_m}{m} \right), \quad (4.5.15)$$

$$\bar{f}^{(5)}(z)|_{z=0} = -\frac{5}{2\pi^4} \sum_{m=1}^n \left(\frac{A_m}{m^4} - 2\pi^2 \frac{A_m}{m^2} - 3\pi \frac{B_m}{m^3} + \pi^3 \frac{B_m}{m} \right), \quad (4.5.16)$$

$$\bar{f}^{(6)}(z)|_{z=0} = \frac{3i}{2\pi^5} \sum_{m=1}^n \left(-15\pi \frac{A_m}{m^4} + 5\pi^3 \frac{A_m}{m^2} - 15\frac{B_m}{m^5} + 10\pi^2 \frac{B_m}{m^3} - 2\pi^4 \frac{B_m}{m} \right). \quad (4.5.17)$$

We want all the derivatives up to n^{th} order to vanish. From (4.5.12) to (4.5.17), we see that if n is even, then the following sufficient, but not necessary, conditions must be satisfied:

$$\sum_{m=1}^n \frac{A_m}{m^j} = 0, \forall j \in \{2, 4, \dots, n-2\}, \quad (4.5.18)$$

and

$$\sum_{m=1}^n \frac{B_m}{m^l} = 0, \forall l \in \{1, 3, \dots, n-1\}. \quad (4.5.19)$$

On the other hand, if n is odd, then the following conditions must be satisfied:

$$\sum_{m=1}^n \frac{A_m}{m^j} = 0, \forall j \in \{2, 4, \dots, n-1\}, \quad (4.5.20)$$

and

$$\sum_{m=1}^n \frac{B_m}{m^l} = 0, \forall l \in \{1, 3, \dots, n-2\}. \quad (4.5.21)$$

Since we mentioned that the function $\cos[2\theta(t, t_f)]$ must be bounded between ± 1 , this imposes constraints on the coefficients A_m and B_m for all times $t \in [0, t_f]$. For $t = 0$, we use $\cos[2\theta(0, t_f)] = 1$ to conclude that

$$\sum_{m=1}^n A_m = 1. \quad (4.5.22)$$

Note that $\cos[2\theta(t_f, t_f)] = 1$ will also be satisfied in this case. We could include constraint equations in equations (4.5.18) to (4.5.21) ensuring that $\cos[2\theta(t, t_f)]$ always has the correct bound, but for the examples we consider below these will not be necessary.

At this stage it is important to realize that there are many types of functions that can satisfy the conditions (4.5.18) to (4.5.22). The procedure will always be the same: We must solve linear equations for the coefficients A_j and B_j that come from (4.5.18) to (4.5.22). However, these will not be sufficient to fix all the coefficients. So we could use the remaining freedom in the coefficients A_j and B_j by inserting the expanded function $\cos[2\theta(t, t_f)]$ into (4.5.11) and minimizing the result.

To see how this works, we will consider an example. To begin, we set $n = 2$. So from (4.5.19) and (4.5.22) we require that

$$B_1 + \frac{1}{2}B_2 = 0, \quad (4.5.23)$$

and

$$A_1 + A_2 = 1. \quad (4.5.24)$$

Using the above results and (4.5.9), we can write

$$\cos[2\theta(t, t_f)] = A_1 \left[\cos\left(\frac{2\pi t}{t_f}\right) - \cos\left(\frac{4\pi t}{t_f}\right) \right] + B_2 \left[\sin\left(\frac{4\pi t}{t_f}\right) - \frac{1}{2} \sin\left(\frac{2\pi t}{t_f}\right) \right] + \cos\left(\frac{4\pi t}{t_f}\right). \quad (4.5.25)$$

One possible approach that we can take is to set $B_2 = 0$. Also, the reader is reminded that for cavity-photon shot noise, (4.5.8) can be written in the time domain as

$$\left\langle T_t \hat{X}(t_f) \hat{X}(t_f) \right\rangle = \left(\frac{g^2}{\Delta} \right)^2 N_{th} (N_{th} + 1) \int_0^{t_f} dt_1 dt_2 \cos[2\theta(t_1, t_f)] \cos[2\theta(t_2, t_f)] e^{-\kappa|t_1 - t_2|}. \quad (4.5.26)$$

Using (4.5.25), (4.5.26) can be calculated and the result will depend on the parameters A_1 and B_2 (the expression is a bit laborious and will not be written here). However, In the short-time limit, we find

$$\left\langle T_t \hat{X}(t_f) \hat{X}(t_f) \right\rangle = \left(\frac{g^2}{\Delta} \right)^2 N_{th} (N_{th} + 1) \frac{\kappa (1 - 2A_1 + 5A_1^2 + 2B_2^2) t_f^3}{16\pi^2} + \mathcal{O}(t_f^4). \quad (4.5.27)$$

Since the B_2 coefficient only adds a positive contribution, we clearly see that choosing $B_2 = 0$ was the best choice. Note that the above term cannot be made to vanish since the equation $1 - 2A_1 + 5A_1^2 = 0$ has imaginary solutions. However, we can find the coefficient such that $f(A_1) = 1 - 2A_1 + 5A_1^2$ is as small as possible. The answer is

$$A_1 = \frac{1}{5}. \quad (4.5.28)$$

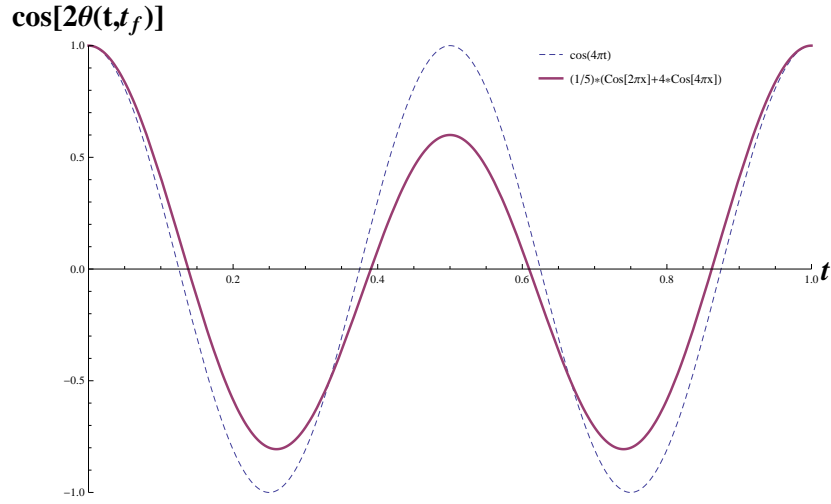


FIGURE 4.5.1. Non-trivial path for $n = 2$
Plot of the path described by (4.5.29) keeping only the second harmonic in the Fourier expansion. We chose $t_f = 1$.

Consequently, for the class of functions where $B_1 = B_2 = 0$, the optimal path to get the best fidelity for short times is given by

$$\cos[2\theta(t, t_f)] = \frac{1}{5} \left[\cos\left(\frac{2\pi t}{t_f}\right) + 4 \cos\left(\frac{4\pi t}{t_f}\right) \right]. \quad (4.5.29)$$

It is also worth mentioning that the function given by (4.5.29) is bounded between ± 1 and so can accurately describe a state transfer protocol.

It is also worthwhile drawing a plot that compares the previous solution we took ($\cos[2\theta(t, t_f)] = \cos\left(\frac{4\pi t}{t_f}\right)$) to the one found in (4.5.29).

Our optimization procedure applied to the case where $n = 2$ gives rise to a non-trivial path, in the sense that the state is never fully transferred to $|g_2\rangle$ (see figure (4.5.1)). To understand this, we follow the noise correlation argument that we gave in the paragraph below (3.2.1). There we argued that it is most advantageous to do the adiabatic evolution in such a way to spend as much time as possible with the state (2.2.26) being in the equal superposition $|g_1\rangle + |g_2\rangle$. This means that it would be favorable to avoid having the state (2.2.26) being all $|g_2\rangle$ so as to maximise the presence of simultaneous $\delta\hat{\omega}_1$ and $\delta\hat{\omega}_2$ noise.

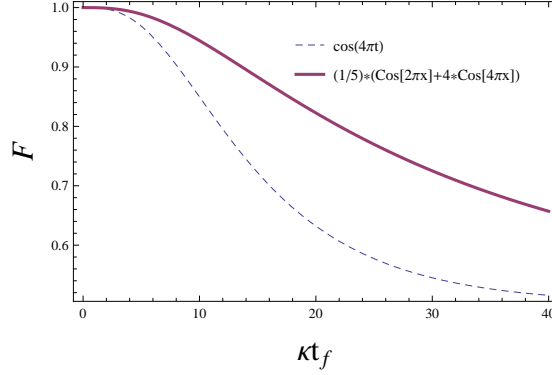


FIGURE 4.5.2. Optimal path comparison

Plot of the fidelity as a function of t_f comparing a circular evolution to the path found in (4.5.29). We have chosen the values $\alpha = \beta = \frac{1}{2}$, $N = 1$, $\frac{g^2}{\Delta} = 0.1$ and $\kappa = 1$. Note that we also turned off the phase $\phi(t)$ so that the closed system Berry's phase ($\gamma_d(t_f)$) vanishes.

Increasing n , it is expected that similar behavior will be exhibited where optimal solutions would differ from the usual linear paths ($\theta(t) = \frac{n\pi t}{t_f}$) that we considered throughout most of this thesis.

As a last note, since for both the paths $\cos[2\theta(t, t_f)] = \cos\left(\frac{4\pi t}{t_f}\right)$ and $\cos[2\theta(t, t_f)] = \frac{1}{5} \left[\cos\left(\frac{2\pi t}{t_f}\right) + 4 \cos\left(\frac{4\pi t}{t_f}\right) \right]$, $\langle \tilde{\Pi}_{0d}(t) \rangle \sim e^{-(t/\tau)^3}$ in the short time limit $\kappa t_f \ll 1$, we can compute the ratio of the decay times for both paths. Defining $\tau_{n=2}$ as the decay time for the path $\cos[2\theta(t, t_f)] = \cos\left(\frac{4\pi t}{t_f}\right)$ and τ_{opt} as the decay time for the path in (4.5.29), we can use (4.3.34) and (4.5.27) to show that

$$\frac{\tau_{opt}}{\tau_n} = \left(\frac{5}{4}\right)^{1/3} \quad (4.5.30)$$

4.6. Phase gate

Throughout this thesis, the whole idea behind considering our state-transfer protocol was to perform a phase gate. We could have avoided dephasing effects altogether if the four-level atom was kept in a superposition state given by a linear combination of $|g_1\rangle + |g_2\rangle$, since this state is insensitive to the noise. The goal here is as follows: for the closed-system case, we know that when performing our state transfer protocol, the state of interest at time

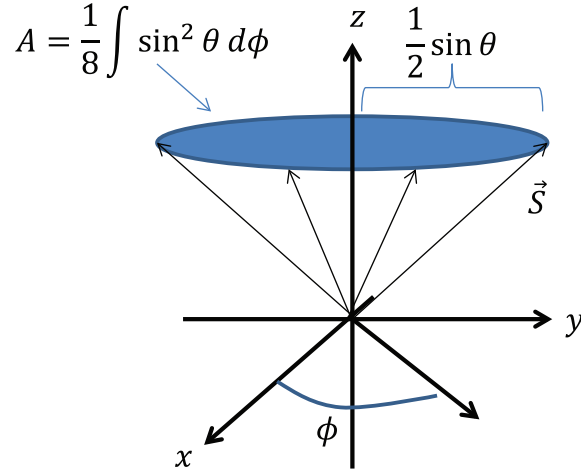


FIGURE 4.6.1. Area enclosed by geometric phase

Figure showing the relation between the area projected onto the x-y plane of a spin-half vector to the geometric phase.

$t_i = 0$ will have the form

$$|\psi(t)\rangle = \alpha |0\rangle + \beta |g_1\rangle, \quad (4.6.1)$$

and that at the end of the state transfer

$$|\psi(t_f)\rangle = \alpha |0\rangle + \beta e^{i\gamma_d(t_f)} |g_1\rangle. \quad (4.6.2)$$

We remind the reader that for a general path $\theta(t)$, the geometric phase is given by

$$\gamma_d(t) = \int_0^t \dot{\phi}(t') \sin^2 \theta(t') dt', \quad (4.6.3)$$

Now suppose that at the end of the state transfer protocol we want $\gamma_d(t_f)$ to be given by a fixed phase α . We can rewrite (4.6.3) as

$$\alpha = \gamma_d(t_f) = \int \sin^2 \theta d\phi. \quad (4.6.4)$$

We can think of (4.6.4) as being proportional to the area projected onto the x-y plane by a spin-half vector \vec{S} executing a closed-path evolution in 3-d space.

In order to see this, we start by writing $\langle \vec{S} \rangle$ in its three dimensional components

$$\begin{aligned} \langle \vec{S} \rangle &= (\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle) \\ &= \frac{1}{2} (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \end{aligned} \quad (4.6.5)$$

The area projected onto the x-y plane is given by

$$dA_{\perp} = \frac{1}{2} \vec{S}_{\text{perp}} dt, \quad (4.6.6)$$

where

$$\vec{S}_{\text{perp}} = \left| \vec{S}_{\perp} \times \dot{\vec{S}}_{\perp} \right| \quad (4.6.7)$$

Then a straightforward calculation shows that

$$\left| \vec{S}_{\perp} \times \dot{\vec{S}}_{\perp} \right| = \frac{1}{4} \dot{\phi} \sin^2 \theta. \quad (4.6.8)$$

Thus we conclude that

$$A_{\perp} = \frac{1}{8} \int \sin^2 \theta d\phi. \quad (4.6.9)$$

If we identify α with A_{\perp} , (4.6.9) shows that we can choose many paths $\theta(t)$ that give the same area projected onto the x-y plane. When noise is present in our system, there will be dephasing effects which will reduce the fidelity of our state transfer protocol (as was shown in (4.3.31)) but it will not modify the geometric phase. Consequently, the idea will be to find an optimal path for a given cutoff frequency (as was done using the short-time expansion protocol of the previous section) in order to minimize dephasing effects. The optimal solution will then fix the function $\theta(t)$. Thus to obtain the desired phase $\alpha = \gamma_d(t_f)$ at the end of the state transfer protocol, the function $\phi(t)$ is chosen such that the integral in (4.6.3) reduces to α . To make things clearer, we give two examples that illustrate the general procedure.

For $n = 1$ in (4.5.9), then it is easy to check that the optimal solution which minimizes $\langle T_\tau \hat{X}^2(t_f) \rangle$ for short times is given by

$$\cos[2\theta(t, t_f)] = \cos\left(\frac{2\pi t}{t_f}\right). \quad (4.6.10)$$

If we choose the function $\phi(t)$ to satisfy

$$\dot{\phi}(t) = ct, \quad (4.6.11)$$

which is simply a linear function of time with slope c , the the integral in (4.6.3) is easy to compute allowing us to fix the parameter c to be

$$c = \frac{4\alpha}{t_f^2}. \quad (4.6.12)$$

Consequently, we are left with

$$\phi(t) = \frac{2\alpha}{t_f^2} t^2, \quad (4.6.13)$$

where we choose the constant such that at $t = 0$, $\phi(0) = 0$. Thus, the function $\phi(t)$ chosen in (4.6.13) will ensure that we get the desired phase at the end of the state transfer protocol when $\theta(t)$ is given by (4.6.10).

For $n = 2$ in (4.5.9), we showed that the optimal solution was given by (4.5.29). With the same quadratic function for the phase $\phi(t)$ as in (4.6.11), the integral in (4.6.3) is now found to be

$$\begin{aligned} \alpha = \gamma_d(t) &= c \int_0^t t \sin^2 \left\{ \frac{1}{2} \arccos \left[\frac{1}{5} \left[\cos\left(\frac{2\pi t}{t_f}\right) + 4 \cos\left(\frac{4\pi t}{t_f}\right) \right] \right] \right\} dt' \\ &= \frac{ct_f^2}{4}. \end{aligned} \quad (4.6.14)$$

Just as in (4.6.12), we conclude that

$$c = \frac{4\alpha}{t_f^2}. \quad (4.6.15)$$

Thus although the path is different, the area of the integral (4.6.3) is the same for both paths where $n = 1$ and $n = 2$ which means that we can use the same function $\phi(t)$ in both cases to obtain the desired phase during the state transfer protocol. Note however that we

could choose a different functional form for the phase $\phi(t)$ which could also be well suited for performing a phase gate. For example, if we had chosen the simple constant

$$\dot{\phi}(t) = c, \quad (4.6.16)$$

then the integral of (4.6.3) would evaluate to be

$$\begin{aligned} \alpha = \gamma_d(t) &= c \int_0^t \sin^2 \left\{ \frac{1}{2} \arccos \left[\frac{1}{5} \left[\cos \left(\frac{2\pi t'}{t_f} \right) + 4 \cos \left(\frac{4\pi t'}{t_f} \right) \right] \right] \right\} dt' \\ &= \frac{ct_f}{2}. \end{aligned} \quad (4.6.17)$$

In this case we would have

$$\phi(t) = \frac{2\alpha}{t_f} t. \quad (4.6.18)$$

In general, after finding the optimal path for a particular value of the frequency cutoff, we are free to choose a convenient functional dependence for the phase $\phi(t)$ such that the integral in (4.6.3) is as simple as possible. As was shown above, for the case where $n = 1$ or $n = 2$, linear or quadratic functions of time are convenient choices.

4.7. Summary

We considered a system where a four-level atom was coupled to a cavity being driven by two coherent drives (each with its own detuning frequency). These created transitions between the atomic states allowing us to perform our state transfer protocol. In order to avoid off-resonant terms, we required that $|\Delta_i| \gg G$. To get the desired Hamiltonian that would allow us to calculate dephasing effects, we first wrote our Hamiltonian in a superadiabatic basis and then performed a Schrieffer-Wolff transformation. In doing so, table (4.1.2) gave a list of conditions that needs to be satisfied in order for our theory to be valid. The Schrieffer-Wolff transformation enabled us to get the appropriate noise terms that allowed us to apply the methods of chapter 3 to calculate the fidelity given by Eq. (4.3.31) which depended on the number of cycles for the state-transfer protocol. The plot of figure (4.4.1) clearly showed that performing the state-transfer protocol over many cycles extended the coherence time of the atomic qubit state. It also demonstrated that

the lifetime of the qubit was greatly enhanced by performing the state-transfer protocol versus leaving the system in its qubit state. It was also noted that one had to be careful not to break the adiabatic criteria by choosing an arbitrarily large value for the number of state-transfer cycles. Using known experimental values for the cavity and drive parameters, the plot of figure (4.4.2) showed the same behavior as (4.4.1) thus enforcing the idea that our state-transfer protocol is a viable way of fighting decoherence effects. By increasing the value of the detuning frequency, figure (4.4.4) showed that the fidelity remains closer to unity for longer times.

We addressed the issue of finding an optimal path for our state-transfer protocol. Using a short-time expansion method along with a class of functions described by Eq. (4.5.9), we saw that for $n \geq 2$, a linear combination of cosine and sine functions with coefficients determined by the equations of (4.5.18) to (4.5.21), and by ensuring that $\cos[2\theta(t, t_f)]$ is appropriately normalized, gave the optimal path. Finally, we showed how one could choose a phase $\phi(t)$ to perform a phase gate. Choosing (4.6.13) would allow us to get the desired geometric phase (using the path obtained in (4.5.29)) but that other convenient choices were also possible, such as in (4.6.18). One very important remark is that we performed a phase gate by going from the state (4.6.1) to (4.6.2) and so for this particular state-transfer protocol, dephasing effects are inevitable.

CHAPTER 5

Conclusion

In conclusion, we developed theoretical methods for calculating dephasing effects when a four-level system (following our state-transfer protocol) is coupled to a quantum dissipative environment. The first step is to write the Hamiltonian in a superadiabatic basis by going into a rotating frame with the appropriate unitary transformation given by Eq. (2.3.3). Then, we performed a secular approximation which amounts to dropping all the off-diagonal terms of the superadiabatic Hamiltonian. This is valid as long as the conditions $\frac{\dot{\theta}}{G} \ll 1$, $\frac{\dot{\phi}}{G} \ll 1$ and $\frac{|\delta\dot{\omega}_i|}{G} \ll 1$ are satisfied. With a purely diagonal Hamiltonian, the phase of the “0d” component of the density matrix was obtained by iterating its equation of motion. This particular component of the density matrix contains all the phase information that is relevant to the state-transfer scheme that was proposed and thus contains the information relevant for dephasing effects. However, our methods can be used to obtain the phase information for any component of the density matrix. In section (3.1), we showed that when the spectral density of the bath degrees of freedom obey a Lorentzian peaked at a non-zero frequency (Eq.(3.2.14)), the phase of the relevant component of the density matrix would come back to unity at the end of the state-transfer protocol. This is only true if the frequency is given by $\nu_0 = \frac{2n\pi}{t_f}$ $\{n \neq 2\}$. Consequently, if it is possible to find a mechanism such that this frequency can be controlled to take on this particular value, it would always be possible to reduce the dephasing effects of having the system coupled to a quantum dissipative bath (if damping is involved, then choosing $\nu_0 = \frac{2n\pi}{t_f}$ $\{n \neq 2\}$ only gives rise to partial recurrences).

After developing these general theoretical methods, we applied them to the case of a four-level atom coupled to a driven cavity. Two laser tones were used to drive the cavity, each detuned from the cavity frequency. The classical component of the laser fields were

used to create transitions allowing us to perform the state-transfer protocol described in chapter three. In order to ensure that each laser field only induced transitions between the appropriate atomic levels (eliminating the off-resonant terms), the conditions $\frac{\alpha_2(t)g_1}{|\Delta_2 - \Delta_1|} \ll 1$ and $\frac{\alpha_1(t)g_2}{|\Delta_2 - \Delta_1|} \ll 1$ needed to be satisfied. Since the cavity interacted with its environment leading to fluctuations in the number of photons inside the cavity, this induced dephasing/dissipation effects when performing our state-transfer protocol. We rewrote the initial Hamiltonian of Eq.(4.1.1) by first writing it in a superadiabatic basis and then performing a Schrieffer-Wolff transformation. Doing so allowed us to apply our theoretical methods to the state-transfer protocol arising from driving the cavity. For our theory to apply, we needed that the conditions in table (4.1.2) to be satisfied. We calculated the fidelity of the state-transfer protocol using a circular path in $\{\Omega_1, \Omega_2\}$ space (with $\theta(t)$ being a linear function of time) allowing the freedom to repeat the state-transfer cycle an arbitrary number of times. We showed that doing so would extend the coherence time of the qubit state being transferred.

To ensure that the adiabatic criteria remained satisfied for this state-transfer cycle, the condition $\frac{n\pi}{Gt_f} \ll 1$ must be satisfied. Finally, for the class of functions considered in Eq.(4.5.9), we gave an algorithm for finding an optimal path that minimized dephasing effects for our particular state-transfer protocol. We considered an example where we chose the frequency cutoff at $n = 2$ and in this case the optimal path was given by (4.5.29). The algorithm was found using a short-time expansion after writing the function $\cos 2\theta(t, t_f)$ in a Fourier series expansion with a finite-frequency cutoff. As the figure in (4.5.2) shows, there is a significant improvement of the fidelity for the path given by (4.5.29) over the function $\cos 2\theta(t, t_f) = \cos\left(\frac{4\pi nt}{t_f}\right)$.

Finally, for the optimal paths found in section (4.4), we were able to find a simple functional form for the phase $\phi(t)$ given by (4.6.13) and (4.6.18) which was particularly well suited for performing a phase gate. We showed that many convenient choices were possible for the phase $\phi(t)$ once the optimal path of $\cos 2\theta(t, t_f)$ had been found.

Our work goes beyond the usual Bloch-Redfield master equation approach for studying state-transfer protocols and performing phase gates as in [29, 48] by performing a secular

approximation in the superadiabatic basis. This ensures that our results are not constrained to environments with short correlation times compared to the coherence time of the “ $0d$ ” component of the density matrix and we don’t need to assume weak dissipation. We also considered a particular noise model arising from photon cavity shot noise instead of adding them by hand which is crucial when performing state-transfer protocols/phase gates for an atom coupled to a cavity.

APPENDIX A

Finding $H_{env}(t)$ for a driven cavity

In section (2.4), we considered a cavity driven by an external field. Making a distinction between the internal cavity modes and the external bath modes, we wrote the Hamiltonian as

$$\hat{H} = \hat{H}_{sys} + \hat{H}_{bath} + \hat{H}_{int}. \quad (\text{A.0.1})$$

The bath Hamiltonian was described by a collection of harmonic modes

$$\hat{H}_{bath} = \sum_q \hbar \omega_q \hat{b}_q^\dagger \hat{b}_q, \quad (\text{A.0.2})$$

which obeyed the commutation relations

$$[\hat{b}_q, \hat{b}_{q'}^\dagger] = \delta_{q,q'}. \quad (\text{A.0.3})$$

Within a rotating wave approximation and using (2.4.15), the coupling Hamiltonian was described as

$$\hat{H}_{int} = -i\hbar \sqrt{\frac{\kappa}{2\pi\rho}} \sum_q [\hat{a}^\dagger \hat{b}_q - \hat{b}_q^\dagger \hat{a}]. \quad (\text{A.0.4})$$

Now, we consider the situation of section (4.1) where an atom is coupled to a driven cavity. Starting at a time $t = t_0 < 0$, we turn on the laser drive tones. In this situation, the density matrix for the atom and environment (cavity + bath) can be written as a product state

$$\hat{\rho}(t_0) = \hat{\rho}_{at}(t_0) \otimes \hat{\rho}_{env}, \quad (\text{A.0.5})$$

with the environmental part to the density matrix being in thermal equilibrium

$$\hat{\rho}_{env} = \frac{1}{Z} e^{-\beta \hat{H}_{env}}. \quad (\text{A.0.6})$$

The Hamiltonian \hat{H}_{env} will incorporate the dynamics of the cavity and bath modes as well as the cavity-bath coupling contributions. Consequently, we can write

$$\hat{H}_{env} = \sum_q \hbar \omega_q \hat{b}_q^\dagger \hat{b}_q + \omega_c \hat{a}^\dagger \hat{a} - i \hbar \sqrt{\frac{\kappa}{2\pi\rho}} \sum_q \left[\hat{a}^\dagger \hat{b}_q - \hat{b}_q^\dagger \hat{a} \right]. \quad (\text{A.0.7})$$

After turning on the laser drive tones, the laser will populate the environment modes so that $\hat{\rho}_{env}(t)$ will no longer be in thermal equilibrium described by (A.0.6) for $t > t_0$. Defining

$$\beta(t) \equiv \beta_1(t) + \beta_2(t), \quad (\text{A.0.8})$$

with the input field being a coherent drive with a classical and quantum part (see (4.1.7)), the environment can approximately be described by a coherent state with a non-zero expectation value for the input field given by

$$\langle \hat{b}_{in}(t) \rangle = \text{Tr} \left\{ \hat{b}_{in} \hat{\rho}_{env}(t) \right\} = e^{i\omega_c t} \beta(t), \quad (\text{A.0.9})$$

where, following (2.4.16), we have

$$\hat{b}_{in} = \frac{1}{\sqrt{2\pi\rho}} \sum_q \hat{b}_q. \quad (\text{A.0.10})$$

We can get a time-independent $\hat{\rho}_{env}(t)$ by performing a displacement transformation using the unitary operator

$$\hat{U}_\beta(t) = e^{-ie^{i\omega_c t} \beta(t) \hat{p}}, \quad (\text{A.0.11})$$

where

$$\hat{p} = \alpha \left(\hat{b}_{in} - i \hat{b}_{in}^\dagger \right), \quad (\text{A.0.12})$$

and α is a proportionality constant. Indeed, one can verify that the unitary operator written in (A.0.11) displaces \hat{b}_{in} in the following way

$$\hat{U}_\beta(t) \hat{b}_{in} \hat{U}_\beta^\dagger(t) = e^{i\omega_c t} \beta(t) + \hat{b}_{in}. \quad (\text{A.0.13})$$

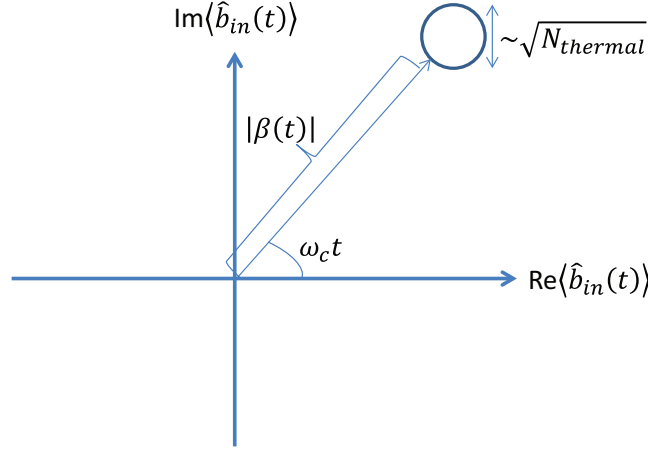


FIGURE A.0.1. Phase space plot of the coherent state $\langle \hat{b}_{in}(t) \rangle$. The above figure corresponds to the phase space plot of the coherent state $\langle \hat{b}_{in}(t) \rangle$. The blob represents vacuum noise arising from thermal fluctuations in $\hat{b}_{in}(t)$ with the amplitude being determined by $\sqrt{N_{th}}$.

With this displacement transformation, $\hat{\rho}_{env}$ transforms as

$$\hat{U}_\beta(t) \hat{\rho}_{env}(t) \hat{U}_\beta^\dagger(t) = \hat{\rho}_\beta \quad (\text{A.0.14})$$

which is now time-independent. The Hamiltonian (A.0.7) will also transform under the unitary operator (A.0.11) as

$$\hat{H}'_{env}(t) = \hat{U}_\beta(t) \hat{b}_{in} \hat{U}_\beta^\dagger(t) - i \hat{U}_\beta(t) \dot{\hat{U}}_\beta^\dagger(t). \quad (\text{A.0.15})$$

Choosing the complex amplitude $\beta(t)$ to solve the classical equation of motion and taking $t_0 \rightarrow -\infty$, the transformed Hamiltonian (A.0.15) becomes

$$\hat{H}'_{env}(t) = \sum_q \hbar \omega_q \hat{b}_q^\dagger \hat{b}_q + \omega_c \hat{a}^\dagger \hat{a} + \left(-i \hbar \sqrt{\kappa} \sum_q e^{i \omega_c t} \beta(t) \hat{a} + h.c. \right) - i \hbar \sqrt{\frac{\kappa}{2\pi\rho}} \sum_q \left[\hat{a}^\dagger \hat{b}_q - \hat{b}_q^\dagger \hat{a} \right] + const \quad (\text{A.0.16})$$

where $-i \hbar \sqrt{\kappa} \sum_q e^{i \omega_c t} \beta(t) \hat{a} + h.c.$ corresponds to the classical drive term.

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